The Hong Kong University of
Science and Technology

Lecture Notes on
Reinforced Concrete Design
[CIVL 3320]

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Declaration

This compilation of lecture notes has been carefully assembled solely for educational purposes, with the primary aim of supporting the students of the CIVL3320 course in their efforts to study and master the course material. **These lecture notes are still under construction.** This early draft probably contains typos, errors, or missing sections, and the reader's assistance in identifying such issues would be greatly appreciated.

Elias DIMITRAKOPOULOS March 11, 2026

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a_v	Shear span.
A_s	Area of longitudinal steel (in tension).
A'_s	Area of compression steel required in a doubly reinforced section.
A_{sv}	Area of shear reinforcement in the form of closed links (per unit spacing s_v).
b	Width of the beam cross-section.
b_v	Breadth of the section effective in shear (implied in $v = \frac{V}{b_v d}$).
b_w	Web width.
d	Effective depth of the section (implied in $v = \frac{V}{b_v d}$).
d'	Distance to the centroid of the compression steel.
f_{cu}	Characteristic compressive strength of concrete.
f_y	Characteristic strength of the longitudinal reinforcement.
f_{yv}	Characteristic strength of the shear reinforcement (links).
g_k	Dead load per unit length, including the self-weight of the beam.
h	Overall height of the beam cross-section.
K	Dimensionless bending moment, defined as $\frac{M}{bd^2 f_{cu}}$.
K'	Maximum allowable dimensionless bending moment for a singly reinforced section.
K_A	Anchorage length factor (HKCC2013 [7], Table 8.4).
l_b	Anchorage length in tension for reinforcement bars.
l_e	Effective anchorage length of a standard 90° bend.
M	Bending moment (general notation).
$M_{AB,\max}$	Maximum bending moment at mid-span of section AB.
$M_{B,\max}$	Maximum bending moment at support B.
$M_{b,\max}$	Maximum bending moment in the beam (used in preliminary design).
ϕ	Diameter of the reinforcement bar.
q	Uniformly distributed load (used in shear force calculation, $V_d = V_q - qd$).
r	Radius of the bend in an anchorage.
s_v	Spacing of the shear reinforcement links.
T	Torsional moment due to ultimate load.

v	Direct shear stress, defined as $\frac{V}{b_v d}$.
v_c	Design concrete shear stress.
v_r	Shear stress from minimum shear links.
v_t	Torsional shear stress.
$v_{t\text{min}}$	Minimum torsional capacity of the concrete section.
v_{tu}	Ultimate torsional shear stress, defined as $\min\left(0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}\right)$.
v_{max}	Maximum shear stress, defined as $\frac{V}{b_v d}$.
V	Shear force (implied in the expression for direct shear stress $v = \frac{V}{b_v d}$).
V_c	Shear force at the center of the support.
V_d	Shear force at the critical section (at a distance d from the support).
V_q	Shear force due to the applied load (used in $V_d = V_q - qd$).
V_s	Shear force at the face of the support.
x	Depth to the neutral axis in a doubly reinforced section.
x_1	Larger center-to-center dimension of the links (used in reinforcement calculations).
y_1	Smaller center-to-center dimension of the links (used in small section conditions and reinforcement calculations).
z	Lever arm distance in a singly reinforced section.
z_{max}	Maximum allowable lever arm.
z_{min}	Minimum allowable lever arm.
γ_m	Material partial safety factor (used in shear capacity calculation, v_c).
s	Depth of stress block.
ϵ_{cu}	Ultimate concrete strain.
ϵ_s	Strain in tension steel.
ϵ'_s	Strain in compression steel.
ϵ_y	Yield strain of steel.
M_u	Ultimate moment capacity.
F_{cc}	Concrete compressive force.
F_{st}	Steel tensile force.
F_{sc}	Steel compressive force.
ρ	Reinforcement ratio (A_s/bh).
ρ_{max}	Maximum reinforcement ratio.
ρ_{min}	Minimum reinforcement ratio.
β	Bond coefficient in Table 5.1, but also
β	End restraint conditions coefficient in Table 8.2.
β	This symbol is also used to introduced many coefficients in the HKCC2013 code...

β_b	Moment redistribution factor
β_f	Factor involved in the calculation of the maximum moment resistance of a flanged section Eq.2.20
P	Applied load.
P_y	Load at yielding.
P_u	Ultimate load.
L	Span length.
Δ	Displacement.
μ	Ductility ratio (Δ_u/Δ_y).

Chapter 1

Basis of design

Overview: This Chapter covers the following topics:

- Focus of the course, common challenges from the perspective of students.
- Fundamental concepts of structural engineering (strength, stiffness, ductility).
- Design premise: brittle vs ductile failure.
- Premise and examples of limit states design.

1.1 Motivational example and fundamental notions

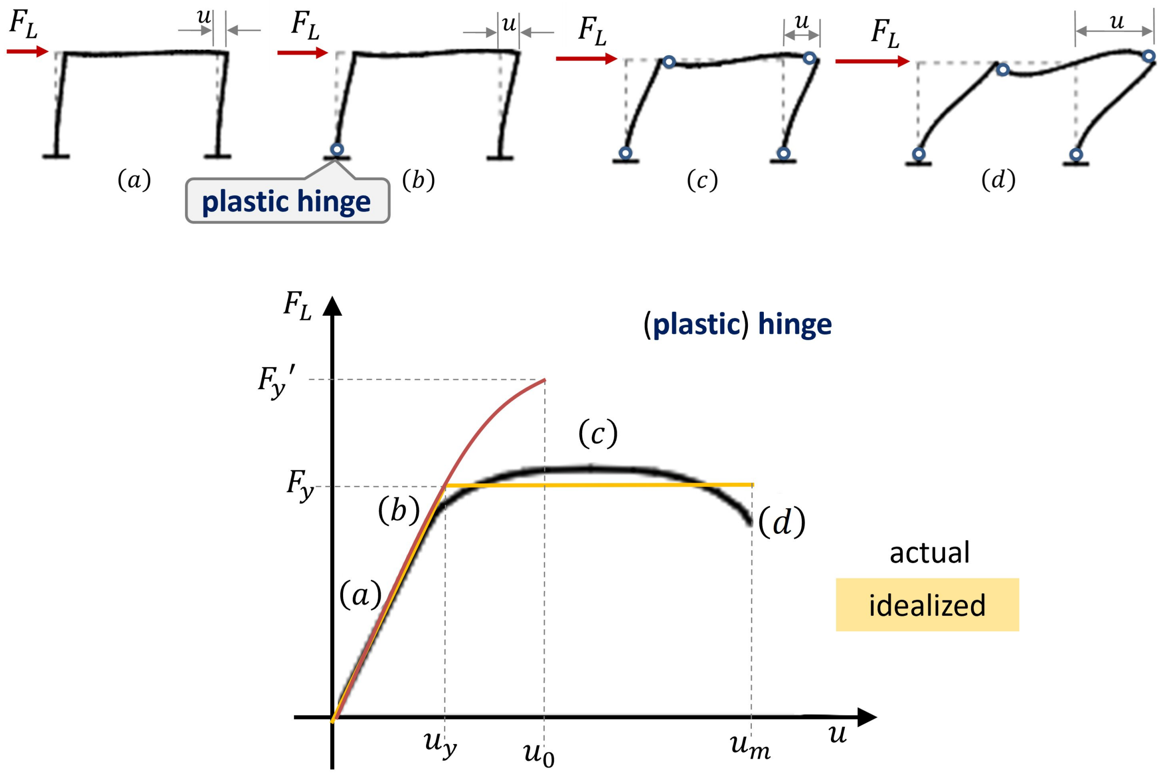

Figure 1.1: Ductile versus brittle structural behavior.

Consider the structural behavior of a typical reinforced concrete portal frame under increasing lateral load F_L. Figure 1.1 illustrates the force - displacement $F_L - u$ behavior of the portal frame and sketches (a-d) a sequence of the pertinent structural systems capturing the behavior of the frame at increasing levels of the lateral load.

Focus first on the behavior of the structure represented by the black curve (Figure 1.1):

- **Elastic Region**: Initially, for low load levels, the frame exhibits linear elastic behavior, with displacement u proportional to load F_L (Figure 1.1a).
- **First Plastic Hinge**: At a critical load level, a critical section in the frame yields (steel yields or concrete crushes), forming a plastic hinge; at that section moment resistance is constant and rotation increases without additional force (Figure 1.1b).
- **Subsequent Hinges**: As the load increases, more sections yield throughout the structure, forming additional plastic hinges (Figure 1.1c). With each additional hinge that forms the slope of the force-displacement curve reduces.
- **Collapse Mechanism**: After a specific number of plastic hinges (which for the examined portal frame is four), the frame becomes a mechanism, unable to resist additional load, the force-displacement curve becomes horizontal and displacement increases at constant load (Figure 1.1d). When the rotation capacity of any of the plastic hinges is exhausted the structure collapses.

The actual force-displacement curve is nonlinear, but it is convenient to **idealize** it as a **bilinear** elastic-plastic (yellow) curve. This simplifies the nonlinearity, as it allows to conventionally define a yield point F_y, u_y —that might not correspond exactly to the first hinge formation at point (b)— before which the behavior is elastic, and after which the behavior sharply changes to plastic (ignoring harderning) see Figure 1.1. In the plastic phase (Figure 1.1c) a constant load (F_y) produces increasing displacement until collapse at u_m, corresponding to the mechanism formation at point (d) of Figure 1.1.

Definitions of basic notions

Key structural properties are derived from the force-displacement curve of a structure (Figure 1.1):

- **Stiffness**: The slope of the elastic portion of the force-displacement curve, representing resistance to deformation.
- **Strength**: The maximum load the structure can sustain, the peak of the curve.
- **Ductility**: The ratio of displacement at collapse (u_m) to displacement at yield (u_y):

$$\mu = \frac{u_m}{u_y} \tag{1.1}$$

Compare now the behavior of the black curve structure with that of the red curve structure. In the case of the red curve structure the load at yielding is higher, but soon after the first yielding, where noticeable decrease of the stiffness takes place, the structure collapses. A fundamental design principle in structural engineering is to avoid brittle failure modes at all costs. Brittle failures are potentially catastrophic because they are sudden and offer little to no warning. Consequently, a black-curve structure with lower strength but greater ductility is preferable to a red-curve structure exhibiting superior strength but brittle behavior as it provides a safer and more predictable response. In this context, ductility provides warning of the imminent failure through tangible signs of plastic deformation, such as excessive cracking.

Given the critical role of ductility in ensuring structural safety, a key question arises: how can a brittle material like concrete be used to create a ductile structure? A first short answer is by prioritizing ductile failure modes over brittle ones. This can be accomplished by ensuring that concrete failure modes (which are typically brittle) are either avoided or delayed until ductile steel failure modes, such as yielding, occur. To achieve this, we must carefully analyze and design the critical cross-sections of the structure, a process whose importance will become evident as we delve deeper into structural behavior. This realization brings us naturally to another fundamental concept in structural engineering that of hierarchical thinking and abstraction.

1.1.1 Hierarchical Thinking and Abstraction

Another cornerstone of structural engineering is hierarchical thinking across multiple levels of abstraction. Specific levels of abstraction are:

- **Structural System**: At the highest level, a *structure* can be conceptualized as a *structural system*—a drastic simplification since a building structure is a lot more than merely a structural system— that allows for systematic analysis (Figure 1.2).
- **Structural Members**: A structural system is the result of various components working together which we think of as structural members. The next level of abstraction are the individual members that constitute the system, such as beams, columns, and slabs.

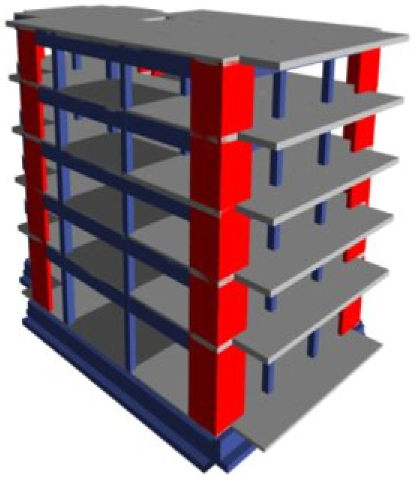

Figure 1.2: Abstraction: A building is modelled as a structural system.

- **Cross-Sections**: At a more detailed level, the analysis zooms into the cross-sections of these members, where material properties, geometry, and reinforcement details dictate the behavior under load.

This hierarchical framework enables to model and analyze structures across distinct levels (Figure 1.3) each providing critical insights into the overall performance. The three levels of abstraction: *structural system*, *structural member*, *cross-section* are not the only. For instance, a more granular level of abstraction, such as the material point, is frequently utilized to analyze stresses in greater detail. Nonetheless, effective design relies on mastering analysis across at least the three most commonly used levels of abstraction. This is imperative to ensure ductile behavior at critical cross-sections, which ultimately produces the desired ductility on the level of members and of the structure.

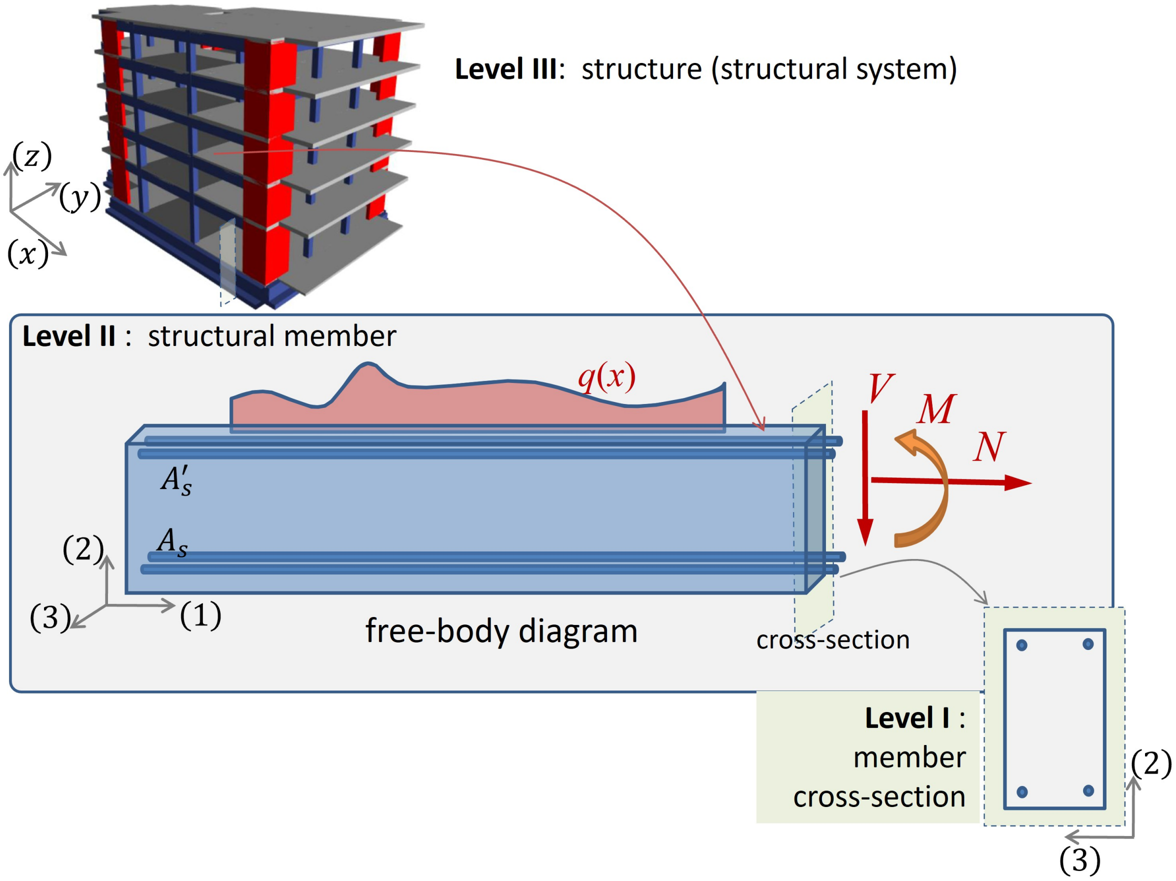

Figure 1.3: Hierarchical abstraction illustration with reference to a typical R/C building: from a structural system to the cross-section level.

1.2 Structural Engineering Background

The study of reinforced concrete design does not exist in isolation It follows a natural progression within a broader sequence of structural engineering knowledge. This sequence typically entails foundational courses such as *Statics*, *Mechanics of Materials*, and *Structural Analysis*. These prerequisite courses establish the essential background required to understand and apply the principles of reinforced concrete design effectively.

It is often the case that, a significant portion of the challenges students encounter in a reinforced concrete design course stems not from the subject itself, but from gaps in their understanding of these foundational topics. Concepts such as equilibrium, stress-strain relationships, and internal forces diagrams—all introduced in the aforementioned courses—are critical for mastering reinforced concrete design. While we will revisit some of these fundamental concepts to bridge potential gaps, the scope of such a review will be limited due to time constraints. Students are therefore encouraged to solidify their understanding of these prerequisites to fully engage with the material in this course.

1.2.1 Motivation: Four-Point Bending Test

The four-point bending test (Figure 1.4) is an archetypal experimental method for evaluating the flexural behavior and load-carrying capacity of reinforced concrete beams. In this setup, a simply supported beam is subjected to two symmetrically applied loads within the middle third of its span. This configuration induces pure bending in the central region, enabling a precise analysis of stress distribution and crack formation.

At very low load levels (P), the stress distribution across the beam's cross-sections remains linear. As the applied load increases incrementally, cracks appear even at modest loads, in the beam's tension zone (Figure 1.4 right). This is because of concrete's low tensile strength. These cracks typically initiate at the bottom face, where tensile stresses peak, and propagate as the load approaches the beam's ultimate capacity, defined by its maximum load-carrying limit. In unreinforced concrete beams, failure occurs immediately upon cracking. In contrast, in reinforced concrete beams steel reinforcement develops tensile forces post-cracking, providing sufficient moment resistance to maintain structural integrity. It is important to realize that steel reinforcement in concrete is needed exactly where cracks develop, and in particular, should be aligned so to bridge those cracks.

Crack Patterns

To understand the observed crack patterns and interpret the beam's behavior at failure, we can draw on core structural engineering principles:

- **Statics & Structural Analysis**: Internal force diagrams, including bending moment (M), shear force (V), and axial force (N).
- **Mechanics of Materials**: Stress distribution, material failure, and the onset of cracking.
- **Reinforced Concrete (RC) Design**: Selection of cross-section and determination of required steel reinforcement area.

In the middle third of the beam, where pure bending occurs ($N(x) = 0$, $V(x) = 0$, $V(x) \neq 0$), a uniform moment generates a linear stress profile—tension at the bottom and compression at the top—with no shear stresses present (see point 2 in Figure 1.4). Consequently, in this region, the stress distribution induces horizontal tensile forces in the bottom portion of the beam. Given concrete's limited tensile strength—roughly $10\,\%$ of its

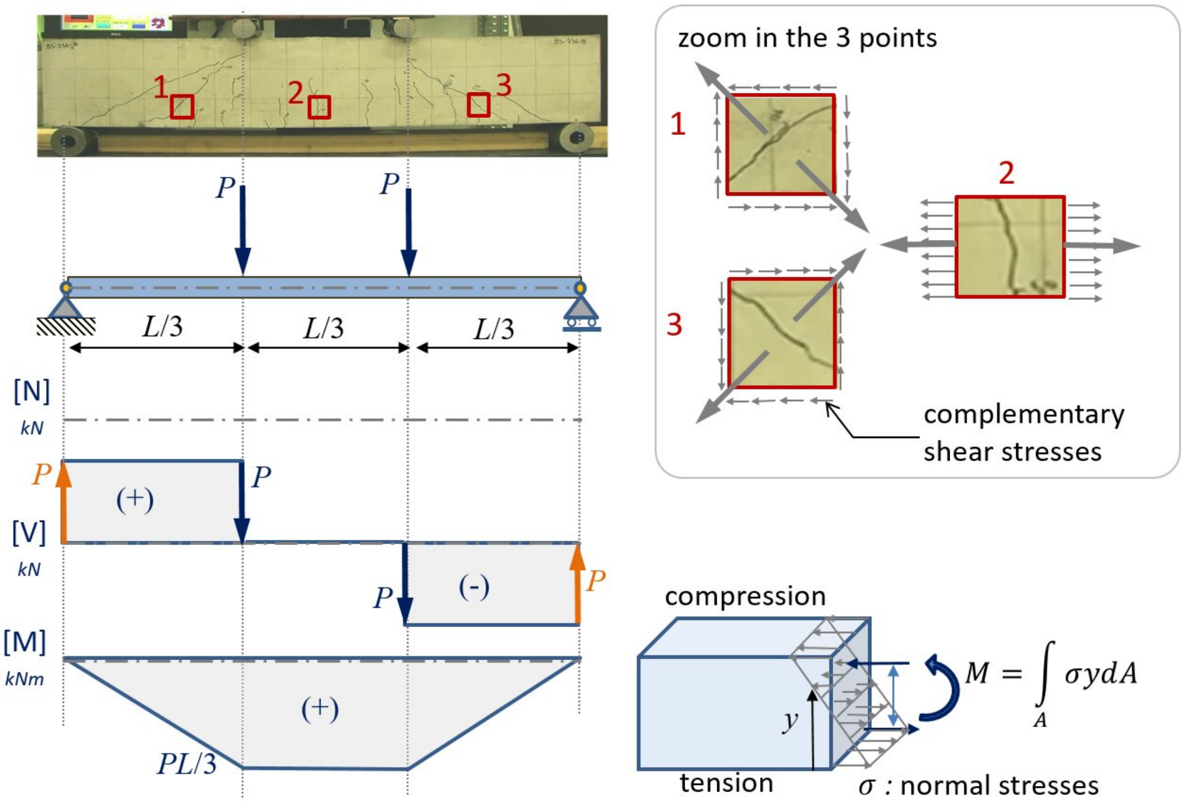

Figure 1.4: Four-point bending test: Experimental motivation.

compressive capacity—it fails in tension along the vertical axis, resulting in the formation of vertical cracks.

In contrast, the outer thirds of the beam are subjected to a combination of bending and shear stresses ($N(x) = 0$, $V(x) \neq 0$, $V(x) \neq 0$). Here, normal stresses—tension below the neutral axis and compression above—rise linearly from zero at the supports to a peak at the load points, coexisting with shear stresses. This interplay between normal and shear stresses gives rise to diagonal cracks, oriented from bottom-left to top-right on the left side (point 1 in Figure 1.4) and from bottom-right to top-left on the right (point 3 in Figure 1.4). These cracks, typically inclined at approximately 45°, signal diagonal tension failure.

To arrive at this understanding, we utilized skills you developed in earlier courses: **Statics** and **Structural Analysis** to determine the internal forces diagrams and specify the axial forces, shear forces and bending moments acting on the beam (at points 1, 2, and 3 (see Figure 1.4). Next, we drew on **Mechanics of Materials**, another foundational prerequisite course, to translate these forces into stress distributions, showing how they drive the vertical cracks in the middle third and the diagonal cracks in the outer thirds, elucidating the beam's behavior. In applying **Reinforced Concrete Design**, we will use such insights to select an appropriate cross-section and reinforcement layout. This example demonstrates how prior structural engineering knowledge is indispensable for understanding design decisions.

1.3 Introduction to Reinforced Concrete

Concrete is a composite material formed by mixing sand, gravel, crushed rock, or other aggregates with a paste of cement and water, creating a rock-like mass. Like most rock-like substances, concrete exhibits high compressive strength but very low tensile strength (see Table 1.1). To overcome this limitation, *reinforced concrete* (R/C) combines concrete with steel reinforcement, which provides the necessary tensile strength—a characteristic property of steel (Table 1.1). Without steel reinforcement, most concrete buildings and structures would not be feasible.

Reinforced concrete is a dominant structural material in engineering construction, widely used globally in applications such as buildings, bridges, water tanks, and dams. R/C structures are composed of interconnected *structural members*—such as beams, slabs, columns, shear walls, and beam-column joints—that work together to sustain applied loads. These members can be formed into diverse shapes and sizes, offering versatility in design, particularly for in-situ R/C, compared to other materials like steel or timber see e.g., Table 1.2.

property	concrete	steel
strength in tension	poor	good
strength in compression	good	good, but slender bars will buckle
strength in shear	fair	good
durability	good	corrodes if unprotected
fire resistance	good	poor

Table 1.1: Comparison of Material Properties: Concrete vs. Steel

Structural Prop.	R/C, P/C	Structural Steel	Timber	Masonry
strength	excellent	excellent	fair	good except in tension
durability	excellent	poor vs. corrosion (unless protected)	poor (unless protected)	excellent
appearance	fair	fair	excellent	excellent
safety	excellent	poor fire resist. (unless protected)	good	excellent
speed of erection	slow for in-situ	very fast	very fast	very fast, labour intensive
on-site versatility	excellent for in-situ RC, poor otherwise	poor	fair	very good

Table 1.2: Comparison of Structural Properties Across Materials.
R/C = Reinforced Concrete, P/C = Prestressed Concrete.

The synergy between concrete and steel in R/C not only enhances strength but also improves overall performance, with R/C exhibiting excellent durability and safety compared to steel structures, which suffer from poor corrosion resistance and fire resistance (Table 1.2 and Table 1.1). However, R/C construction can be slower for in-situ applications, a trade-off for its adaptability (Table 1.2). This combination of adaptability, strength, and durability makes reinforced concrete a cornerstone of modern engineering.

The effectiveness of reinforced concrete also stems from the thermomechanical compatibility between concrete and steel. Their similar coefficients of thermal expansion—approximately $(7 \sim 12) \cdot 10^{-6}$ per $^{\circ}$C for concrete and $10 \cdot 10^{-6}$ per $^{\circ}$C for steel—ensure minimal relative movement during temperature changes, preventing internal stresses. Additionally, a strong bond at the interface, particularly with ribbed steel bars, ensures efficient load transfer between the materials. This compatibility, combined with the complementary mechanical properties shown in Table 1.1, allows concrete and steel to work together seamlessly, enhancing the overall performance of R/C structures.

1.4 Building Codes and Design Standards

Building codes, often referred to as design codes are state or municipal bylaws that establish the minimum requirements for the design, construction, and maintenance of building structures. Bylaws are formal rules and regulations established by an organization, association, or governmental body to govern its operations and management. In the context of building codes, bylaws outline the legal framework for enforcing design and construction standards.

1.4.1 British Standard (British Code)

BS 8110 Structural Use of Concrete: British Standards for the design and construction of R/C (reinforced concrete) and P/C (prestressed concrete) structures. This code is based on limit state design principles. Used for most civil engineering and building structures, bridges and water-retaining structures are covered by separate standards (BS 5400 and BS 8007) [13].

- 1st version → published in 1985 (BS 8110:1985)
- 2nd version of Part 1 (BS 8110-1:1997) → published in 1997
- 31 Mar 2010, BSI (British Standards Institution) withdrew the UK Standards which conflict with the Eurocodes as required under the agreement between National Standards Bodies and CEN (European Committee for Standardization)
- BS 8110 was superseded by Eurocode 2 (BS EN 1992), though parts of the code have been retained in the National Annex of the Eurocode

1.4.2 Eurocode 2 (European Code)

Eurocode 2 (EC2) is the abbreviation for EN 1992 EC2 [3]: (Design of Concrete Structures). EC2 covers the design of buildings and civil engineering works constructed in plain, R/C, prestressed, and precast concrete. **BS EN 1992** (e.g. BS EN 1992-1-1:2004) refers to the **Eurocode 2** standard (EN 1992) as officially adopted and published by the British Standards Institution (BSI) for use in the United Kingdom. It is not a separate, purely national British standard, but the harmonised European Eurocode document issued under the British prefix, often supplemented by the UK National Annex (**BS NA EN 1992-1-1**). In UK and Hong Kong practice, references to “EC2” or “Eurocode 2” typically mean this BS EN version together with its National Annex.

- BS EN 1992-1-1:2004, Part 1-1: General rules and rules for buildings [3]
- BS EN 1992-1-2:2004, Part 1-2: General rules – structural fire design [4]
- BS EN 1992-2:2005, Part 2: Concrete bridges – design and detailing rules [5]
- BS EN 1992-3:2006, Part 3: Liquid retaining and containment structures [6]

1.4.3 American code

- ACI 318 Building Code Requirements for Structural Concrete and Commentary [1]

1.4.4 Chinese code

- GB 50010-2010 Code for design of concrete structures [15]

1.4.5 Hong Kong building regulations (by Buildings Department)

In December 2011, the HKSAR Government Development Bureau decided to migrate from British standards to Eurocodes for the design of public works civil engineering structures – mandatory adoption of Eurocodes in Hong Kong commences in 2015. The HKCC2013 code [7] is primarily based on the British Standard BS8110:1985 (Parts 1 and 2), and the two codes are thus similar.

- **Code of Practice for Structural Use of Concrete 2013** (Hong Kong Concrete Code, HKCC2013 [7]): 2020 edition available at [PDF]
- Building (Construction) Regulations 1990 [Link]

The course materials are written to conform to the Hong Kong Concrete Code (HKCC2013) [7], while BS 8110 and EC2 are also referred when they are appropriate.

1.5 Limit States Design

Rational structural design aims to ensure that a structure remains safe under severe loading conditions while maintaining its appearance, durability, and performance during normal operation. To achieve this, modern design codes adopt the *Limit States Design* (LSD) framework, which addresses both safety and functionality under statistical uncertainty [3, 1]. This approach focuses on two primary types of limit states:

- **Ultimate Limit State (ULS)**: Ensures the safety of people and the structure by preventing collapse or failure.
- **Serviceability Limit State (SLS)**: Ensures functionality, user comfort, and aesthetic integrity (e.g., by controlling deflection and cracking).

The core objective of Limit States Design is to achieve acceptable probabilities that a structure will not become unfit for its intended purpose—that is, it will not exceed a limit state beyond which it fails to meet the relevant design criteria. While additional limit states exist, such as those related to fatigue or fire resistance, this course will primarily focus on ULS, with a secondary emphasis on SLS.

1.5.1 Ultimate Limit States

The *Ultimate Limit State* (**ULS**) addresses conditions that could lead to structural failure, prioritizing the safety of both people and the structure itself. ULS design ensures that neither the entire structure (e.g., Fig 1.5) nor its components (e.g., Figs 1.6, 2.23) will collapse, overturn, or buckle under design ultimate loads. Additionally, ULS should verify that no mechanism forms — that the structure will not transform into a mechanism— unless actions exceed their design values (E_d), where all structural properties are estimated using design values. This is expressed as:

$$E_{d,\text{dst}} < E_{d,\text{stb}}$$

where:

- $E_{d,\text{dst}}$: Destabilizing action effects.
- $E_{d,\text{stb}}$: Stabilizing action effects.

1.5.2 Serviceability Limit States

In contrast, the *Serviceability Limit States* (**SLS**) focus on conditions beyond which the structure fails to meet specified service requirements, affecting its functionality, user comfort, or appearance. For structural concrete, typical SLS considerations include controlling cracking, deflection, and stress levels (particularly in prestressed concrete). Other SLS criteria may involve managing excessive vibration, ensuring fire resistance, or addressing fatigue, though these are less commonly emphasized in this context. SLS verification ensures that:

$$E_d \leq C_d \quad \text{or} \quad E_d \leq R_d$$

where:

- C_d: The nominal value of a functional limit, such as an allowable crack width or deflection.

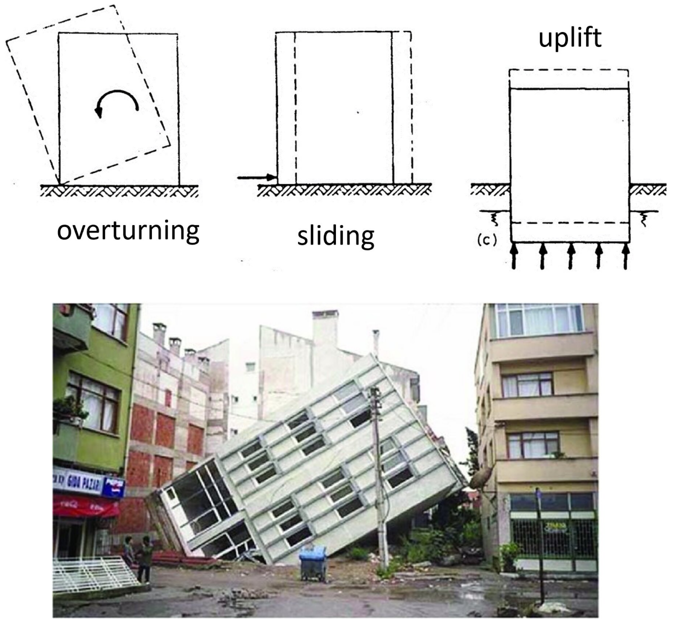

Figure 1.5: Examples of Ultimate Limit State (ULS) failure modes - Overturning. Source DOI: 10.1016/B978-0-12-820513-6.00007-2.

- R_d: An allowable stress at a critical section, often based on material properties like f_{ctk} (the tensile strength of concrete). For SLS, partial safety factors are typically set to unity ($\gamma_c = \gamma_s = 1.0$).

For *Reinforced Concrete* (R/C) members, SLS verification focuses on cracking and deformations. For *Prestressed Concrete* (P/C) members, the emphasis is on stresses under service loading, deformations, and, in some cases, cracking.

Figure 1.6: Examples of Ultimate Limit State (ULS) failure modes - Mechanism. Source DOI: 10.1007/s10518-023-01843-3.

1.6 Characteristic and Design Values

In structural engineering, ensuring safety requires accounting for the inherent variability in loads and material resistances. This variability is addressed through the concepts of *characteristic values* and *design values*, which provide a statistical framework for design under uncertainty. This section explores these concepts, beginning with a motivational example to illustrate the underlying principles, followed by formal definitions and their application in design.

1.6.1 Motivational Example: Tension Testing of a Bar

Consider the tension testing of a steel bar, where the applied load (S) represents the demand, and the bar's resistance (R) represents its capacity. If load and resistance were known deterministic variables, design would be simple: ensure $R > S$ (Figure 1.7 top). Reality is more complicated though. Both loads and resistances are not fixed values but follow statistical distributions, often approximated as normal distributions for simplicity, though more complex distributions may be more realistic. The fundamental requirement for safety is that the resistance must exceed the demand, i.e., $R > S$. However, due to variability, there often exists a region between the two distributions R and S, where failure occurs as the demand exceeds the capacity (red area in Figure 1.7 bottom).

Relying solely on mean values (μ) of load and resistance is insufficient because it ignores variability, represented by the standard deviation (σ). The overlap in the distribution curves (Figure 1.7) indicates the probability of failure, emphasizing the need for a more conservative approach to ensure safety. This is where characteristic and design values come into play, providing a systematic method to account for statistical uncertainty.

1.6.2 Characteristic Values

Characteristic values are statistically derived to represent the extremes of a distribution, ensuring a low probability of exceedance. For material strength (R), the characteristic value R_k is defined as:

$$R_k = \mu_R - 1.64\sigma_R$$

where μ_R and σ_R are the mean and standard deviation of the resistance, respectively. This value corresponds to the 5th percentile, meaning only $5\,\%$ of samples are expected to have a strength below R_k. Conversely, for loads (S), the characteristic value S_k is defined as:

$$S_k = \mu_S + 1.64\sigma_S$$

Here, S_k represents the 95th percentile, where only $5\,\%$ of load samples are expected to exceed this value. These definitions ensure that characteristic values provide a conservative estimate of both strength and load, forming the basis for further safety adjustments.

1.6.3 Design Values

Design values take the conservative approach further by applying partial safety factors to characteristic values, reducing the probability of failure even more. For resistance, the design value R_d is calculated as:

$$R_d = \frac{R_k}{\gamma_m}$$

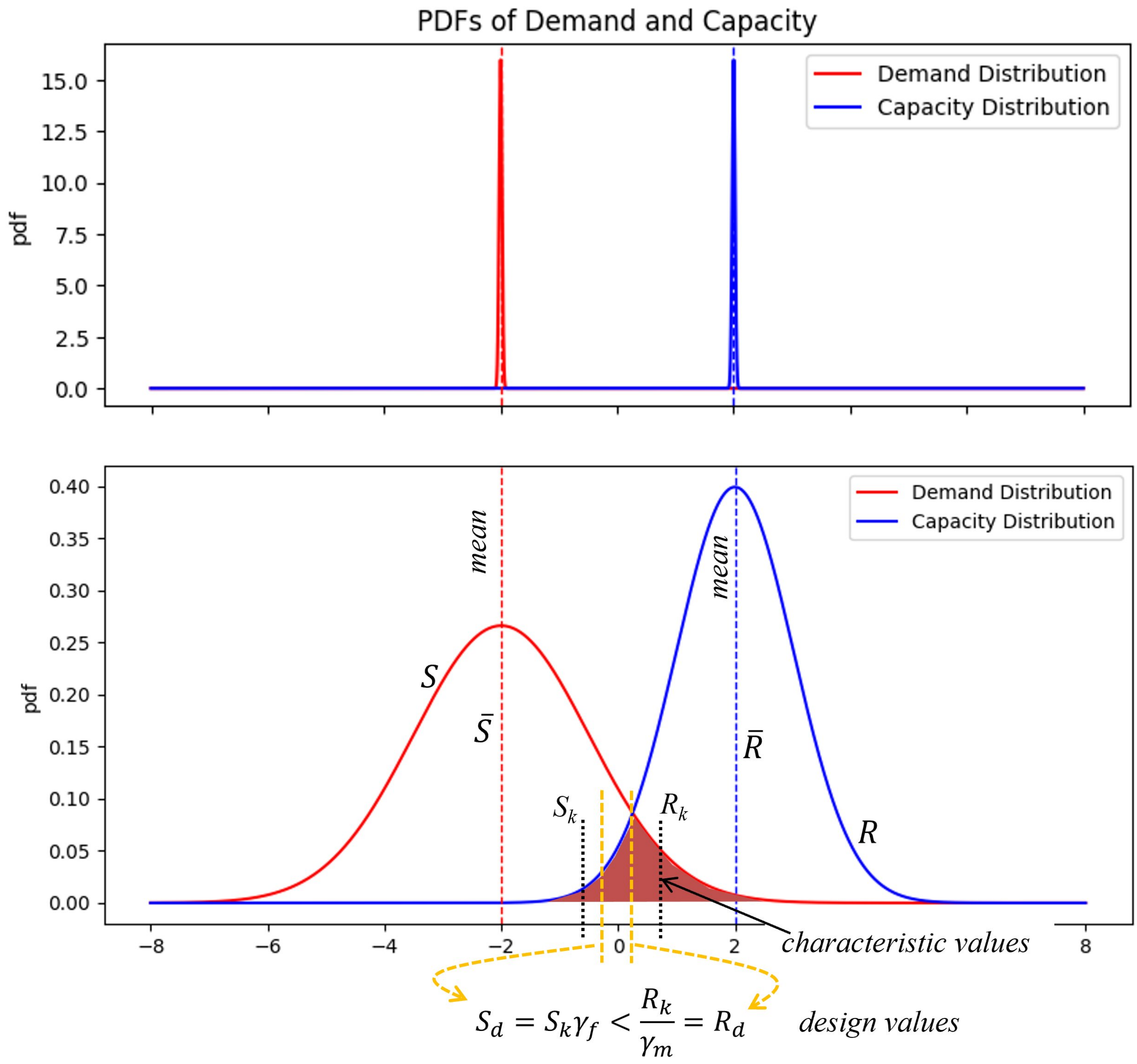

Figure 1.7: Normal distribution curves illustrating the overlap between load (S) and resistance (R) in a tension test, highlighting the failure region where $S > R$.

where $\gamma_m > 1$ is a **partial safety factor** accounting for uncertainties in material properties (e.g., variability in manufacturing or degradation over time). For loads, the design value S_d is:

$$S_d = \gamma_f \cdot S_k$$

where $\gamma_f > 1$ is a partial safety factor reflecting uncertainties in load estimation (e.g., unexpected load increases). The design criterion then ensures safety by requiring:

$$\frac{R_k}{\gamma_m} = R_d > S_d = \gamma_f \cdot S_k$$

This approach reduces the probability of failure without explicitly calculating it, providing a practical and standardized method for structural design. By incorporating statistical variability and safety factors, characteristic and design values ensure that structures remain safe and reliable under a wide range of conditions.

1.7 Material properties needed for R/C design

1.7.1 Concrete Material Properties

Here we outline the key material properties of concrete relevant to structural design.

Compressive Strength

Compressive strength is the primary property used in concrete design, determining its ability to withstand loads. It is measured 28 days after casting using cubic ($150\,\mathrm{mm}$) or cylindrical specimens. In general:

- Hong Kong characterizes concrete grades based on cubic specimen values.
- In both EC2 and ACI codes, all design calculations are based on the cylinder strength f_{ck} (EC2 [11]) and f_c' (ACI [1]), respectively. For EC2, the relationship between the cylinder strength and cube strength $f_{ck}/f_{ck,cube}$ respectively (in $\mathrm{N/mm^2}$) is as follows: 20/25, 25/30, 30/37, 35/45, 40/50, 45/55, 50/60, 60/75, 70/85, 80/95, 90/105
- Cubic specimens yield higher strength than cylindrical ones, for example: C25/30 ($25\,\mathrm{MPa}$ cylindrical, $30\,\mathrm{MPa}$ cubic specimen).
- Normal (strength) concrete: $25\,\mathrm{MPa}$ to $60\,\mathrm{MPa}$; high-strength concrete $> 60\,\mathrm{MPa}$.

(Indirect) Tensile Strength

Tensile strength is much lower than compressive strength, approximately 1/10th of compressive strength. Usually measured indirectly (e.g., via splitting test on cylinders).

- Expressed as (for normal concrete EC2):

$$f_t = \left(\frac{1}{10} \text{ to } \frac{1}{13}\right) f_{cu}$$

- In Eurocode 2 (**EC2**), the characteristic axial tensile strength of concrete, f_{ctk}, is often derived from the characteristic compressive strength, f_{ck}. For the $5\,\%$ fractile value, EC2 provides an approximate relationship:

$$f_{ctk,0.05} = 0.7 \cdot f_{ctm}$$

  where f_{ctm} is the mean axial tensile strength, calculated as:

$$f_{ctm} = 0.3 \cdot f_{ck}^{2/3} \quad (\text{for } f_{ck} \leq 50\,\mathrm{MPa})$$

  or, for higher strengths ($f_{ck} > 50\,\mathrm{MPa}$): $f_{ctm} = 2.12 \cdot \ln\left(1 + \frac{f_{cm}}{10\,\mathrm{MPa}}\right)$, where $f_{cm} = f_{ck} + 8\,\mathrm{MPa}$. The indirect tensile strength (e.g., splitting tensile strength) is closely related to f_{ctk}.

- Expressed as (for normal concrete **ACI**): $f_c' = 0.8 f_{cu}$
- The **ACI 318** code provides empirical relationships to estimate the tensile strength of concrete, which is essential for design considerations like crack control.
- The indirect tensile strength, often approximated through the splitting tensile strength f_{ct}, is related to the specified compressive strength f_c':

$$f_{ct} = 0.56 \cdot \sqrt{f_c'} \quad (\text{in } \mathrm{MPa})$$

  alternatively, the modulus of rupture f_r, which reflects tensile strength in bending, is given by:

$$f_r = 0.62 \cdot \sqrt{f_c'} \quad (\text{in } \mathrm{MPa})$$

These values are used to estimate the concrete's tensile capacity for design purposes.

Strength Development

The strength of concrete increases over time, a critical factor influencing construction scheduling and long-term structural performance. This development is most pronounced in the first week, with significant gains in compressive strength, before stabilizing around 28 days. The 28-day compressive strength serves as the standard benchmark for assessing concrete quality in design and construction. A typical strength development curve for concrete made with Class 42.5 Portland cement, achieving a 28-day compressive strength of $30\,\mathrm{MPa}$, is illustrated in Figure 1.8. This curve can be mathematically modeled using the mean compressive strength at time t, denoted $f_{cm}(t)$, which is expressed as:

$$f_{cm}(t) = \beta_{cc}(t)\, f_{cm}$$

where f_{cm} is the mean compressive strength at 28 days, and $\beta_{cc}(t)$ is a time-dependent coefficient given by:

$$\beta_{cc}(t) = \exp\left\{ s\left[1 - \left(\frac{28}{t}\right)^{1/2}\right]\right\}$$

Here, t is the age of the concrete in days, and s is a coefficient dependent on the cement type (typically around 0.2 to 0.38 for Portland cement). This model captures the rapid early strength gain and the gradual stabilization observed in Figure 1.8.

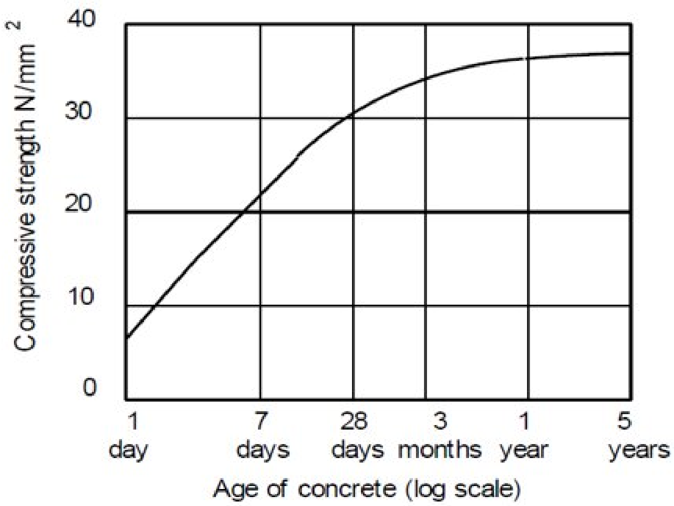

Figure 1.8: Typical strength development curve for concrete made with Class 42.5 Portland cement, showing a 28-day compressive strength of $30\,\mathrm{MPa}$.

Modulus of Elasticity and Poisson's Ratio of Concrete

The modulus of elasticity reflects concrete's stiffness, a key parameter for deformation calculations. The short-term static modulus of elasticity of concrete (HKCC2013 [7]) is:

$$E_c = 3.46\sqrt{f_{cu}} + 3.21$$

where:

- f_{cu}: Concrete cube compressive strength.

Property	Values										
f_{cu} ($\mathrm{N/mm^2}$)	20	25	30	35	40	45	50	55	60	65	70
E_c ($\mathrm{kN/mm^2}$)	18.7	20.5	22.2	23.7	25.1	26.4	27.7	28.9	30.0	31.1	32.2

Table 1.3: Design Values of Elastic Modulus for Normal-Weight Concrete

Property	Values							
$f_{ck}/f_{ck,cube}$ ($\mathrm{N/mm^2}$)	20/25	25/30	30/37	35/45	40/50	45/55	50/60	60/75
E_{cm} ($\mathrm{kN/mm^2}$)	30	31	33	34	35	36	37	39

Table 1.4: Design Values of Mean Elastic Modulus for Concrete

- E_c: Design value of elastic modulus.

Poisson's ratio quantifies the lateral strain response of concrete under axial loading, important for elastic analysis. For design calculations (or where linear elastic analysis is appropriate), Poisson's ratio is usually taken as 0.2.

1.7.2 Steel Material Properties

This subsection details the properties of steel reinforcement, focusing on yield strength.

Yield Strength

The yield strength, denoted as f_y, represents the load at which steel begins to deform plastically, making it a critical parameter for reinforcement bars. This transition from elastic to plastic behavior might be gradual, in which case the yield strength point is defined conventionally (e.g., $0.2\,\%$ offset method).

The two most common strength categories of steel in construction are:

- **High-Yield Bars**-Ribbed Steel Reinforcing Bars (denoted with the letter 'T'):

$$f_y = 500\,\mathrm{N/mm^2} = 500\,\mathrm{MPa}$$

  - 500B – most commonly used.
  - 500C – high ductility, which may be used in seismic design or similar situations.

- **Mild-Steel Bars**-Plain Steel Reinforcing Bars (denoted with the letter 'R'):

$$f_y = 250\,\mathrm{N/mm^2} = 250\,\mathrm{MPa}$$

  Note: Mild-steel bars are now NOT recognised in the European Union and no longer available for general use in the UK and Europe. Modern design primarily uses high-yield ribbed bars.

- For instance, T32 denotes a $32\,\mathrm{mm}$ diameter bar with $500\,\mathrm{MPa}$ strength, and R12 denotes a $12\,\mathrm{mm}$ diameter bar with $250\,\mathrm{MPa}$ strength.

1.7.3 Design Strengths

Design strengths adjust (reduce) characteristic strengths using partial safety factors to ensure safety, as outlined in the HKCC2013 code (see Figure 1.9).

$$\text{design strength} = \frac{\text{characteristic strength}(f_k)}{\text{partial factor of safety}(\gamma_m)} \tag{1.2}$$

where the partial factor of safety for material strength, γ_m, for the ULS are given in HKCC2013 by Table 1.5. The partial factor of safety accounts for variation in material strength, and uncertainties in accuracy of the methods of analysis.

Accordingly, the

- design strength for concrete:

$$\frac{f_{cu}}{\gamma_m} = \frac{f_{cu}}{1.5} = 0.67 f_{cu}$$

- design strength for steel:

$$\frac{f_y}{\gamma_m} = \frac{f_y}{1.15} = 0.87 f_y$$

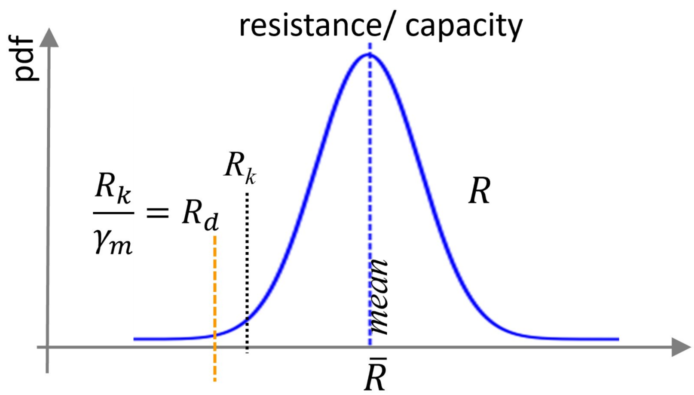

Figure 1.9: Normal Probability Distribution of Strength

Material/Design Consideration	**Value of γ_m**
Steel reinforcement	1.15
Concrete	Flexure or axial load: 1.50
	Shear strength without shear reinforcement: 1.25
	Bond strength: 1.40
	Others (e.g., bearing stress): ≥ 1.50

Table 1.5: Partial Safety Factors for Materials (γ_m)

A Note on Nomenclature: The characteristic compressive strength of concrete is denoted as f_{cu} in HK2003, but as f_{ck} in EC2.

1.8 Loads, Characteristic and Design Values

This section defines the types of loads acting on structures, explains how characteristic loads are determined, and details how they are combined in design to ensure safety under various conditions, as per the HKCC2013 design code.

1.8.1 Dead and Live Loads (F_k)

Characteristic loads (F_k) represent the expected values of loads on structures, categorized based on their nature and variability. These include dead loads (G_k), imposed (live) loads (Q_k), and wind loads (W_k), each requiring specific consideration in design.

Dead Load (G_k)

Dead load (G_k) encompasses the self-weight of the structure, including finishes, ceilings, services, and partitions. In reinforced concrete (R/C) structures, the self-weight constitutes a significant portion of the total dead load. The **specific weight of normal-weight reinforced concrete** varies by region:

- $24.0\,\mathrm{kN/m^3}$ (or $2400\,\mathrm{kg/m^3}$) in the UK and USA;
- $24.5\,\mathrm{kN/m^3}$ (or $2450\,\mathrm{kg/m^3}$) in Hong Kong;
- $25.0\,\mathrm{kN/m^3}$ (or approximately $2500\,\mathrm{kg/m^3} = 2.5\,\mathrm{t/m^3}$) in the Eurocode (EC) and China.

Imposed Load (Live Load) (Q_k)

Imposed load (Q_k) arises from occupancy, such as people, furniture, and equipment. These loads are variable and specified by design codes for different building types. Relevant codes include the Hong Kong Code of Practice for Dead and Imposed Loads 2011 (http://www.bd.gov.hk/english/documents/index_crlist.html) and the Chinese code GB 50009-2012 (Load Code for the Design of Building Structures). In Hong Kong, live loads are conservatively estimated due to high population density, ranging from $2\,\mathrm{kN/m^2}$ for apartments to $5\,\mathrm{kN/m^2}$ for public areas. Table 1.6 provides minimum uniformly distributed live loads for various building types, while Table 1.7 compares minimum live loads across regions.

type of use		min live loads ($\mathrm{kN/m^2}$)
apartment buildings	private units	1.92
	public rooms	4.80
	corridors	3.84
office buildings:	offices	2.40
	lobbies	4.80
	corridors	3.84
	above 1st floor	3.84
stores:	first floor	4.80
	upper floor	3.60
warehouse:	light storage	6.00
	heavy storage	12.00

Table 1.6: Minimum live loads for different types of buildings

case: floor area usage	minimum live load ($\mathrm{kN/m^2}$) HK	UK	China
domestic building	2.5	1.5	1.5
office for general use	3.0	2.5	2.5
shop floor for the display and sale of merchandise	5.0	4.0	3.5
car park (vehicles not exceeding $2500\,\mathrm{kg}$ gross mass)	4.0	2.5	4.0

Table 1.7: Comparison of minimum live loads across regions

Wind Load (W_k)

Wind loads (W_k) are environmental and direction-dependent, impacting structural stability based on exposure and building geometry. These loads require careful consideration in design, particularly for tall or exposed structures.

1.8.2 Characteristic and Design Load Values

Characteristic loads (G_k, Q_k, W_k) are assumed to follow a normal distribution, defined as $\mu \pm 1.64\sigma$, ensuring $95\,\%$ confidence in load estimates, as illustrated in Figure 1.10.

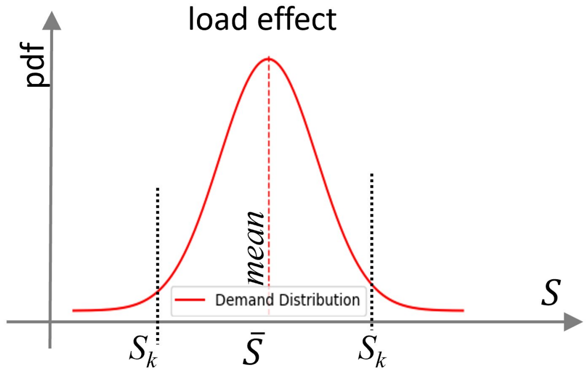

Figure 1.10: Normal Probability Distribution of Load Variability

Design loads are derived by applying **partial safety factors (γ_f)** to characteristic loads to account for uncertainties, including:

- Variations in loads (e.g., unusual increases or unforeseen additional loads);
- Design assumptions and inaccuracies in calculations;
- Unforeseen stress redistributions;
- Constructional inaccuracies.

Table 1.8 reproduces Table 2.1 of HKCC2013, providing partial safety factor values γ_f for different load combinations. The terms "adverse" (adv.) and "beneficial" (bnf.) distinguish between loads that increase the likelihood of failure (e.g., overturning, maximum moment) and those that reduce the likelihood of failure (e.g., stabilizing effects). For example, adverse factors are applied to loads producing a more critical design condition, while beneficial factors (often lower) are used for loads producing a less critical condition. The examples of Section 1.9 elucidate the distinction between adverse and beneficial loads.

load combination	**dead**		**imposed**		**earth and water pressure**	**wind**
	adv.	bnf.	adv.	bnf.		
1. dead and imposed (and earth and water pressure)	1.4	1.0	1.6	0	1.4	–
2. dead and wind (and earth and water pressure)	1.4	1.0	–	–	1.4	1.4
3. dead, imposed and wind (and earth and water pressure)	1.2	1.0	1.2	0	1.2	1.2

Table 1.8: Partial Safety Factors for Loads γ_f

The design load derives from the characteristic (Figure 1.10) with the aid of the partial factor of safety:

$$\text{design load} = \text{characteristic load}\,(F_k) \times \text{partial factor of safety}\,(\gamma_f) \tag{1.3}$$

1.9 Design Load Combinations

Load combinations ensure structures can withstand combined effects. Common design load combinations for ULS include:

1. $1.4G_k + 1.6Q_k$: Dead + live, adverse;
2. $1.4G_k + 1.4W_k$ or $1.0G_k + 1.4W_k$: Dead + wind;
3. $1.2G_k + 1.2Q_k + 1.2W_k$: All combined, with reduced factors.

1.9.1 Example: Design against uplifting

Determine the reaction force at A in order to check uplift in the structure of Figure 1.9.1. According to the sign convention (Figure 1.9.1) a negative reaction force ($R_A < 0$) implies uplifting.
Assume:

- a dead UDL load $20\,\mathrm{kN/m}$ and a live concentrated load of $170\,\mathrm{kN}$ at C,
- characteristic material strengths $f_{cu} = 30\,\mathrm{MPa}$, and $f_y = 500\,\mathrm{MPa}$.

Solution
We need to define the appropriate design load combination. The correct design load combination is specific to the failure scenario examined. In this case we should determine the load arrangement that produces the highest tendency to uplift at point A. If the distributed dead load and the concentrated live load produce a clockwise moment about point B, they tend to uplift the beam at point A and hence they have an adverse effect. If they produce a counterclockwise moment about point B they have a beneficial effect. Accordingly, we assume the pertinent partial factor γ_f values:

- the dead load over AB has a stabilizing beneficial effect (Table 1.8): $\gamma_f = 1.0$
- the dead load over BC has a destabilizing adverse effect (Table 1.8): $\gamma_f = 1.4$
- the live load at point C has a destabilizing adverse effect (Table 1.8): $\gamma_f = 1.6$

Thus, to design against uplifting at point A, the ULS design Load combination is (Figure 1.12). For this load combination, the moment equilibrium about point B gives:

$$\sum M_B = 0 \rightarrow R_A \cdot 6 = \gamma_f(20 \cdot 6 \cdot 3) - \gamma_f(20 \cdot 2 \cdot 1) - \gamma_f(170 \cdot 2) \rightarrow$$

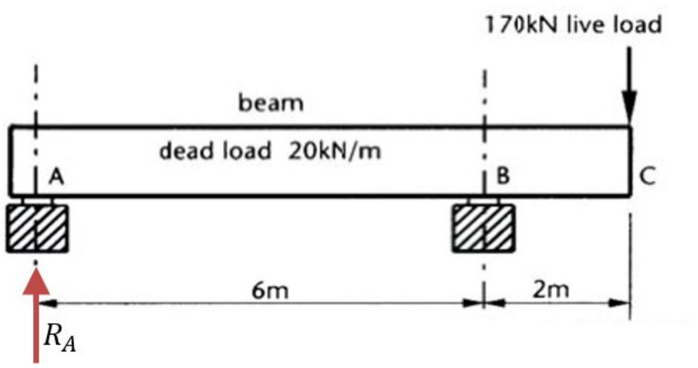

Figure 1.11: Beam supported at A and B. R_A : reaction force at A

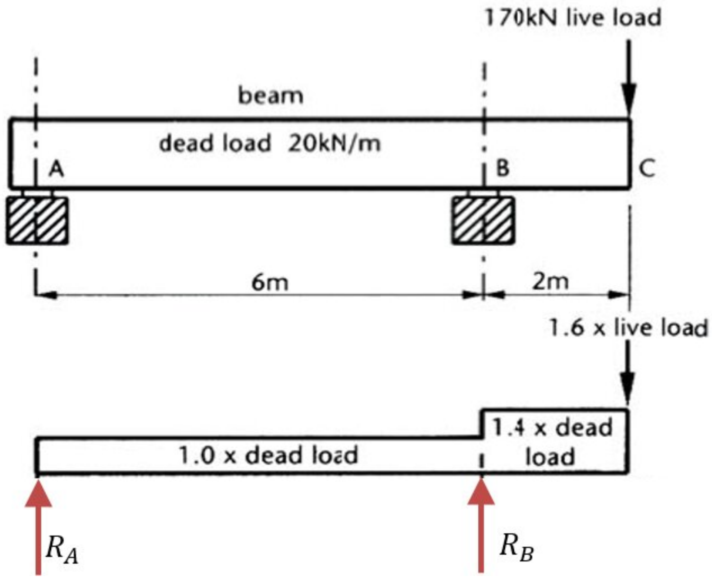

Figure 1.12: Design load combination for ULS for calculating the minimum R_A reaction force at A

$$R_A = -\frac{(272 \cdot 2 + 56 \cdot 1 - 120 \cdot 3)}{6.0} = -40\,\mathrm{kN}$$

Since $R_A < 0$, yes, the beam will try to uplift at point A unless a weight of $40\,\mathrm{kN}$ is added on the beam at point A.

1.9.2 Design Load Combinations for Beams

This section explores the analysis of beams under various loading conditions to determine the critical design moments and forces at the Ultimate Limit State (ULS). To demonstrate, consider the three-span continuous beam with a constant cross-section, supporting a uniformly distributed dead load (including self-weight) of G_k and an imposed load of Q_k as in Figure 1.13. The critical load arrangements for the ULS should capture all maximum positive (sagging) moments in all spans and all minimum negative (hogging) moments at all supports. Taking into account the symmetry of the structure, three load combinations are sufficient to determine all such maximum (in absolute value) bending moments: i.e., all maximum positive (sagging) moments in all spans and all minimum negative (hogging) moments at all supports.

Load Combination for Maximum Span Moments M_{AB} and M_{CD}

To determine the load combination that maximizes the moment in span AB, we utilize the influence line for the maximum moment at the midspan of AB. Note that the maximum moment does not occur exactly at the middle of the span; nevertheless, for the purpose of defining the load combination this inaccuracy is acceptable.

According to the influence line for the midspan moment M_{AB} (see Figure 1.13), M_{AB} increases with loading on spans AB and CD (adverse effect) and decreases with loading on span BC (beneficial effect). Thus, to establish the Ultimate Limit State (ULS) load combination—representing the worst-case scenario—we apply the maximum design load to spans AB and CD and the minimum design load to span BC, using the partial safety factors γ_f from Table 1.8:

- Maximum design load: $1.4G_k + 1.6Q_k$,

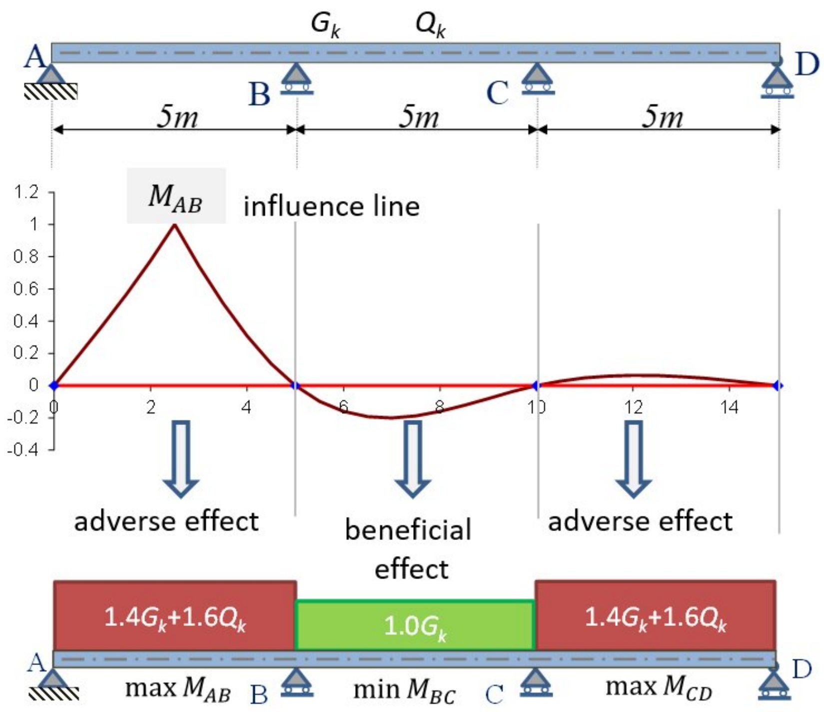

Figure 1.13: Influence line for the midspan moment M_{AB} and the corresponding load combination for maximum moments in spans AB and CD.

- Minimum design load: $1.0G_k + 0.0Q_k$.

  The load combination of Figure 1.13, simultaneously produces:

- The maximum positive moment (M_{max}) in span AB,
- The maximum positive moment (M_{max}) in span CD. This follows from the symmetry between the influence lines of M_{AB} and M_{CD}, where M_{CD} similarly increases with loading on spans AB and CD (adverse effect) and decreases with loading on span BC (beneficial effect). Consequently, the same load combination that maximizes M_{AB} (see Figs. Figure 1.13 and Figure 9.3) also maximizes M_{CD},
- The maximum shear forces and reaction forces at supports A and D, as determined by their respective influence lines (not shown),
- The minimum moment (M_{min}) in span BC, which may be a negative (hogging) moment, based on the influence line of M_{BC} shown in Figure 1.14.

Load Combination for Maximum Span Moment in Span BC

To maximize the span moment in span BC, the influence line for the midspan moment M_{BC} indicates that M_{BC} increases with loading on span BC (adverse effect) and decreases with loading on spans AB and CD (beneficial effect). Therefore, to establish the Ultimate Limit State (ULS) load combination, we apply the maximum design load to span BC and the minimum design load to spans AB and CD, as shown in Figure 1.14.

Load Combination for Minimum Negative Support Moments at Supports B and C

To determine the load combination that produces the minimum negative support moment at B (i.e., the maximum absolute value of the negative moment), we refer to the

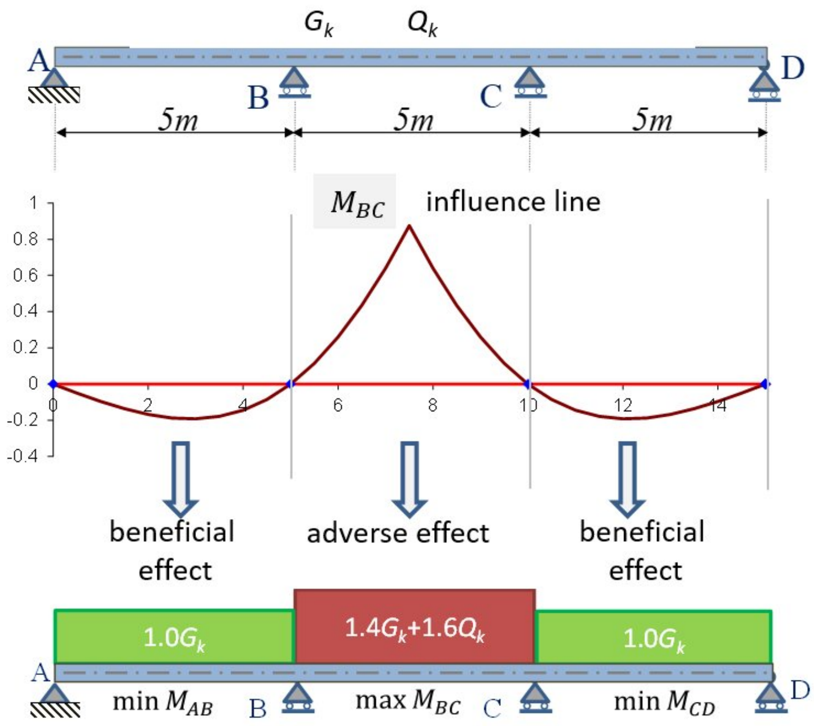

Figure 1.14: Load Combination 2: Bending moment and shear force diagrams for the maximum moment in span BC.

corresponding influence line (Figure 1.15). The negative moment M_B increases with loading on spans AB and BC (adverse effect) and decreases with loading on span CD (beneficial effect). To minimize M_B, we apply the maximum design load to spans AB and BC and the minimum design load to span CD (Figure 1.15).

The load combination in Figure 1.15 simultaneously captures:

- The minimum negative moment (or maximum absolute M_{max}) at support B.
- The minimum negative moment (or maximum absolute M_{max}) at support C, due to symmetry.
- The maximum shear forces and reaction forces at support B, and, due to symmetry, at support C.

Bending Moment Envelope

The bending moment envelope summarizes the maximum moments and forces across the structure (in this case the three-span continuous beam with equal spans), derived from critical load combinations (of Figs 1.13, 1.14, and 1.15). Similarly, the shear force envelope of Figure 9.6 identifies the maximum shear forces critical for shear reinforcement design. Together, these envelopes (Figs. 1.16 and 9.6), encapsulate the worst-case scenarios at the ultimate limit state (ULS) across all spans and supports, serving as a fundamental tool for structural design.

Specifically, the bending moment envelope identifies four critical moments essential for designing the beam:

1. **M_{max} in Spans AB and CD (Maximum Positive Moment in Outer Spans):**
   - *Location*: Near but not exactly at midspan of spans AB and CD.

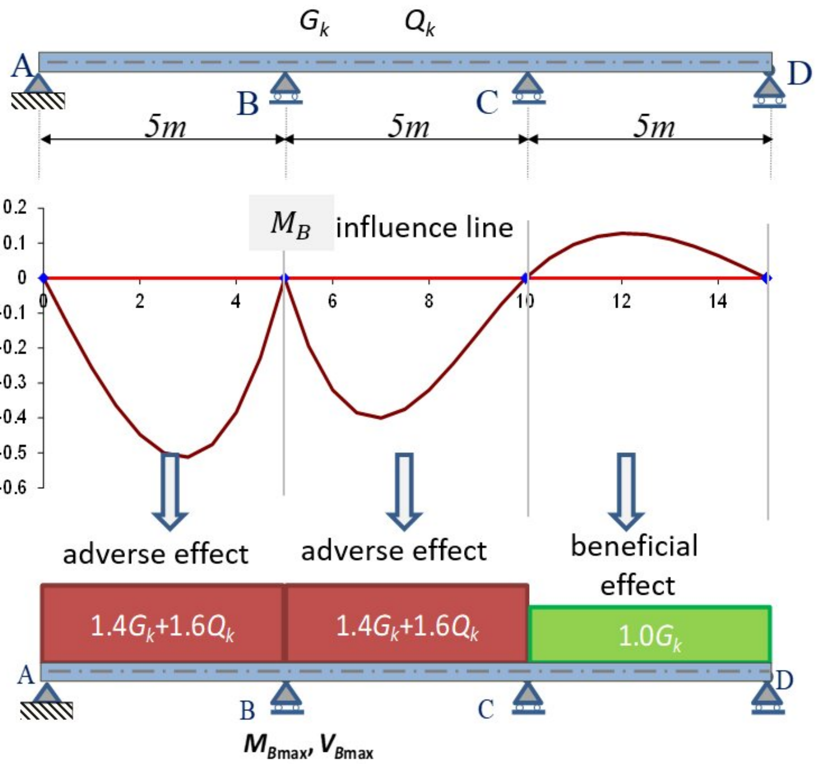

Figure 1.15: Load Combination 3: Bending moment and shear force diagrams for maximum negative moments at supports B and C.

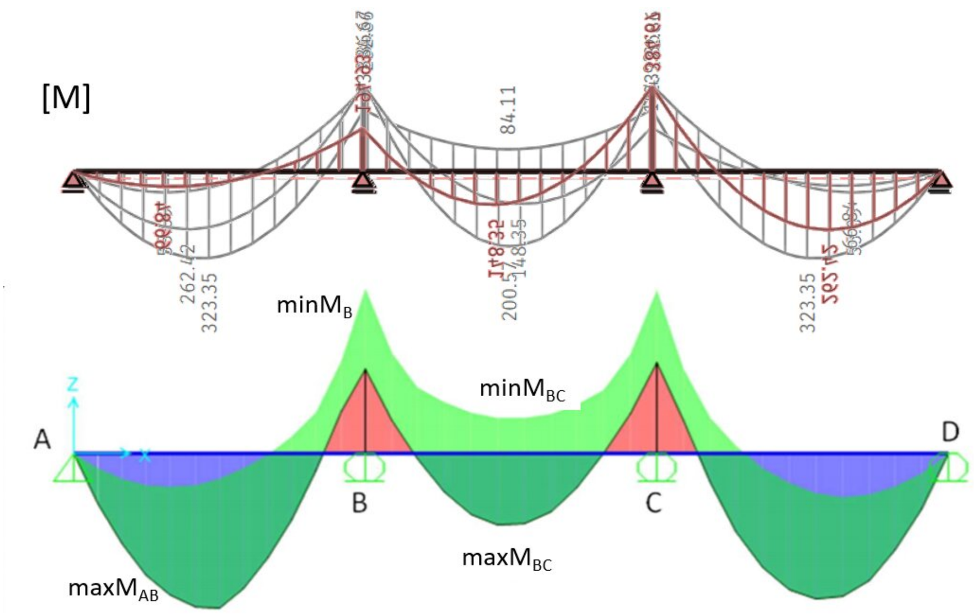

Figure 1.16: Superimposed bending moment diagrams from diffeerent load combinations produce the bending moment envelope for a three-span continuous beam with equal spans.

- *Value*: Determined from Load Combination of Figure 1.13 (Figure 9.3 shows the pertinent bending moment diagram).

2. **M_{max} in Span BC (Maximum Positive Moment in Middle Span):**
   - *Location*: Midspan of span BC.
   - *Value*: Determined from Load Combination of Figure 1.14 (Figure 9.4 shows the pertinent bending moment diagram).

3. **M_{max} at Support B (Maximum Negative Moment at Support B):**
   - *Location*: At support B.
   - *Value*: Determined from Load Combination of Figure 1.15 (Figure 9.5 shows the pertinent bending moment diagram).

4. **M_{max} at Support C (Maximum Negative Moment at Support C):**
   - *Location*: At support C.
   - *Value*: Equal to the maximum negative moment at support B, due to symmetry.

1.9.3 General Rules for Continuous Beams

For continuous beams, imposed loads must be arranged to determine the critical design conditions at all sections of the structure. For continuous beams and slabs in buildings (without cantilevers) subjected to predominantly distributed loading, the following simplified rules may be applied (see Figure 1.17):

- **Maximum moment at supports**: Apply the characteristic permanent load $\gamma_G G_k$ across all spans and the characteristic imposed load $\gamma_Q Q_k$ on the two adjacent spans of the targeted support.
- **Maximum moment in spans**: Apply $\gamma_G G_k$ across all spans and $\gamma_Q Q_k$ on alternate spans including the targeted span.

1.9.4 Shortcut for design values for Continuous Beams

According to HKCC2013> clause 6.1.2.3> Table 6.1 (Table 1.9) may be used to calculate the design ultimate bending moments and shear forces for uniformly loaded continuous beams, subject to the following provisions:

(a) The characteristic imposed load Q_k may not exceed the characteristic dead load G_k.

(b) Loads should be substantially uniformly distributed over three or more spans.

(c) Variations in span length should not exceed $15\,\%$ of the longest span.

Note: No redistribution of the moments calculated from this table should be made.

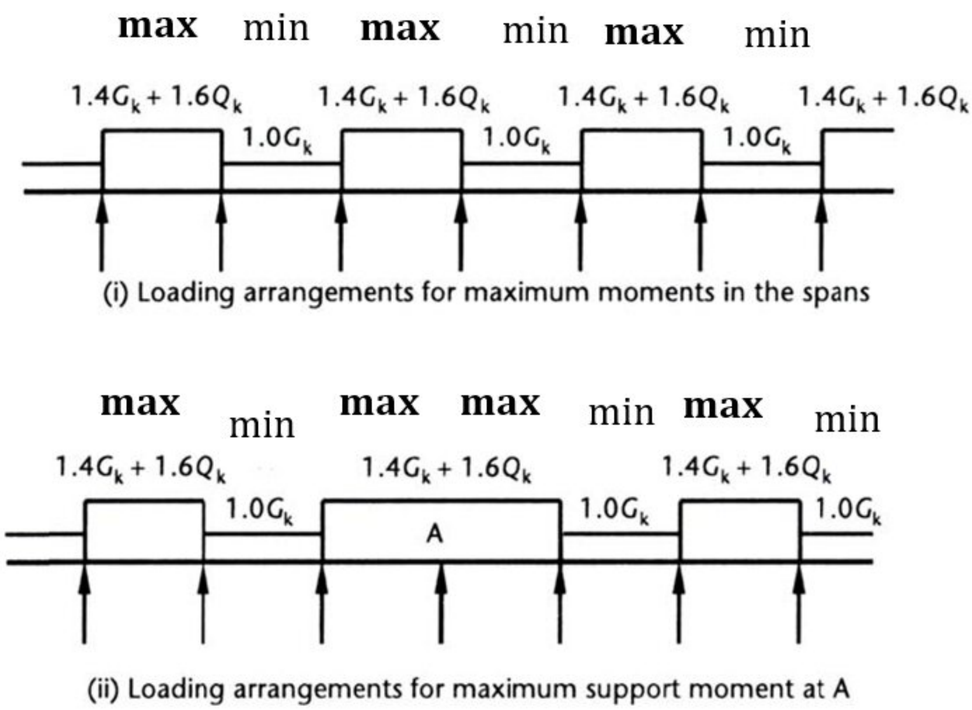

Figure 1.17: Load combination rules for the ULS in multi-span continuous beams.

	At outer support	**Near middle of end span**	**At first interior support**	**At middle of interior spans**	**At interior supports**
Moment	0	$0.09FL$	$-0.11FL$	$0.07FL$	$-0.08FL$
Shear	$0.45F$	–	$0.6F$	–	$0.55F$

Table 1.9: Design ultimate bending moments and shear forces [HKCC2013> clause 6.1.2.3> Table 6.1]

FAQs on Design Principles for Reinforced Concrete

This is a collection of frequently asked questions (FAQs) related to the design principles of reinforced concrete, based on common student inquiries and clarifications provided during lectures.

Q1: Why does the European standard use cylindrical strength while the British Standard uses cubic strength?

This is a convention adopted by the respective codes. Eurocode (EC) actually refers to both (e.g., C20/25, where 20 is cylindrical and 25 is cubic strength in MPa).

Q2: Why do we apply both characteristic strength and a partial safety factor?

The double reduction is intentional and rational. The characteristic strength deals with material property scatter (lab/test variability). The partial safety factor γ_m deals with everything else (model uncertainty, geometry, construction, execution tolerances). This separation is one of the key advances of modern limit state design over older allowable stress methods, rational treatment of uncertainties.

Q3: Is it true that only concrete uses characteristic strength and steel does not?

No, characteristic strength is used for both materials. Refer to **Lecture 1, slides 31–32 for concrete and slide 34 for steel**.

Q4: How do we determine if an effect is adverse or beneficial?

An effect is adverse if the load increases the likelihood of failure (e.g., increases the design moment or reduces stability) and beneficial if it decreases it. For example, consider the vertical reaction at a support. If a permanent load (e.g., the dead weight of a foundation) helps stabilize a structure against uplift caused by wind or pattern loading, it is considered *beneficial* (partial safety factor $\gamma_f = 1.0$). Whether a load is beneficial or adverse can be established by several methods including an appropriate *influence line* at that point.

Q5: What constitutes a "critical design condition" and what loadings create it?

A critical design condition is the combination of loads that produces the most critical design requirement e.g., internal force (shear, moment, etc.) or displacement. To produce such a critical design conditions we combine the concept of partial factors of safety (e.g., $1.4G_k$ and $1.6Q_k$), and load combinations.

Q6: Why don't we apply the most adverse condition (Dead + Live) to all spans for the bending moment envelope?

Loading all spans with $1.4G_k + 1.6Q_k$ (or equivalent) does not necessarily produce the maximum moment at every point (e.g., mid-spans vs. supports). Specific pattern loading is required to find the absolute maximum and minimum values. This is explained in detail in Lecture 3.

Q7: Why does the HK Code stress block use the factor 0.67 in $0.67f_{cu}$?

The 0.67 factor is an empirical coefficient accounting for long-term loading effects (concrete exhibits different strength under sustained loads compared to rapid lab tests) and differences between laboratory conditions and actual site application (e.g., orientation, compaction, and the difference between bending vs compressive strength). Note that this is NOT a safety factor; the partial safety factor $\gamma_c = 1.5$ is applied separately.

Q8: Why is the compressive strength of steel taken as equal to its tensile strength?

In reinforced concrete design, we assume the compressive strength of steel is the same as its tensile strength ($\approx f_y$), provided that the steel bars are properly restrained against buckling by concrete and stirrups.

Q9: What should I do if some load combinations produce positive moments and others produce negative moments at the same section?

The bending moment envelope must show the **extreme results** from all critical combinations. If a section experiences both sagging (positive) and hogging (negative) moments under different loading patterns, y ou must account for both limits in the design envelope to ensure the design covers all possibilities.

Q10: Do we need to know both the BS and HK codes?

The Hong Kong Code is largely based on BS 8110:1985, so they are very similar. The HK Code is required for the course, but references to Eurocodes are made as the HKSAR government has migrated to Eurocodes for public works since 2015.

Q11: Why is the stress block parabolic?

This reflects the non-linear stress-strain behavior of concrete. For design simplicity, the HK code allows replacing this with an equivalent rectangular stress block (refer to **??**).

Chapter 2

Analysis of R/C Cross-sections

Overview: This Chapter covers the following topics:

- Three basic assumptions underpin the of R/C cross-sections:
  plane sections, zero tensile strength, ULS at ε_{cu}.
- Material laws: Parabolic-plastic for concrete, elastic-plastic for steel.
- Design process: Use compatibility, material laws, and equilibrium to find the steel reinforcement required.
- Equivalent stress block: $f_{c,max} = 0.45 f_{cu}$, $s = 0.9x$ (adjustable by f_{cu}).

2.1 A general theory for ultimate flexural strength

2.1.1 Basic Assumptions

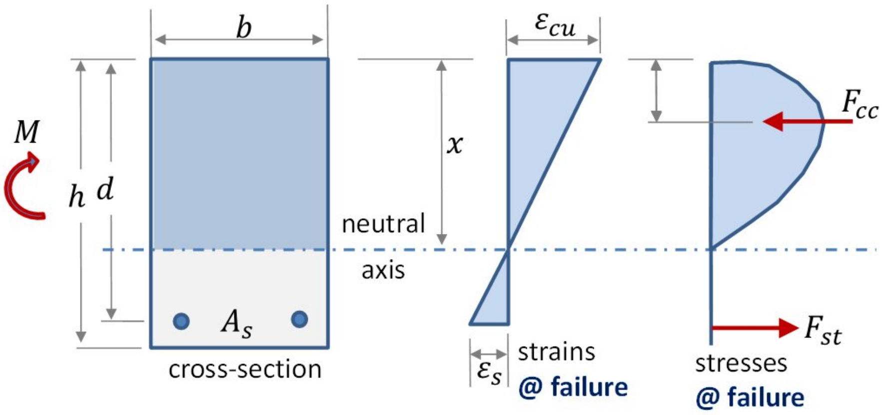

Figure 2.1: ULS: strain and stress distributions at failure

Three assumptions underpin the analysis and design of R/C cross-sections:

- **Assumption 1**: Plane sections remain plane after flexural deformation.
- **Assumption 2**: The Ultimate Limit State (ULS) is reached when the maximum strain in concrete ε_c reaches the limit value $\varepsilon_c = \varepsilon_{cu}$, where:

$$\varepsilon_{cu} = \begin{cases} 0.0035 & \text{if } f_{cu} \leq 60\,\mathrm{N/mm^2} \\ 0.0035 - 0.00006(f_{cu} - 60) & \text{if } f_{cu} > 60\,\mathrm{N/mm^2} \end{cases}$$

- **Assumption 3**: Tensile strength of concrete is ignored ($f_{c,tension} = 0$).

2.1.2 Ramifications of the basic assumptions

- *Assumption 3*: The tensile strength of concrete is not only low, but more critically, it is unreliable, so conservatively we ignore it in design. This implies that any concrete in the tension region of the cross-section does not affect the design equations, and in that sense, it does not exist. In design equations the only force concrete can produce is a compressive force which conventionally we assume as positive $F_{cc} > 0$.
- Assumption 2: ULS failure is defined by concrete strain (specifically crushing at) $\varepsilon = \varepsilon_{cu}$, not steel yielding. At ULS steel might or might not have yielded.
- *Assumption 1*: A direct ramification of Assumption 1 is that the strain distribution in the cross-section is linear (see Figure 2.1). In reality, this linear strain distribution is valid for high span-to-depth ratios ($L \gg h$) but less accurate when $L \approx h$.
- *Assumptions 1 and 2*: Since a straight line is defined with two points, and the strain profile is linear we need only two points to define the strain distribution anywhere in the cross-section (Figure 2.2). One point is given by the definition of the ULS according to *Assumption 2*: at the extreme point of the compression zone the concrete strain is $\varepsilon = \varepsilon_{cu}$

(see Figure 2.1). Usually the second point is the depth of the location of the neutral axis (NA) x where by definition $\varepsilon_{NA} = 0$. Once a second point is found, the linear strain profile is completely defined and the strain at any point of the cross section can be easily calculated. For instance, the strain of tension steel is:

$$\varepsilon_s = \frac{d - x}{x} \varepsilon_{cu}$$

.

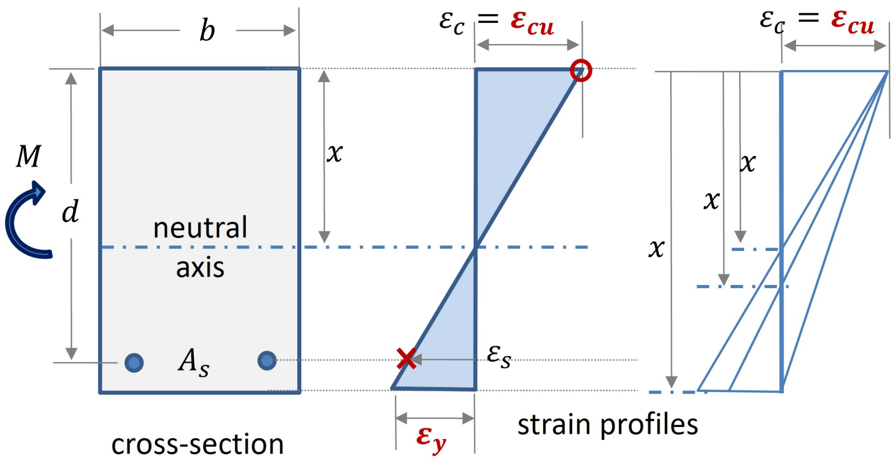

Figure 2.2: ULS: Since the strain profile is linear, it is defined by two points: the extreme concrete strain ε_{cu} at the top, and the neutral axis depth x where $\varepsilon = 0$.

2.2 Constitutive Laws (Stress-Strain Relationships)

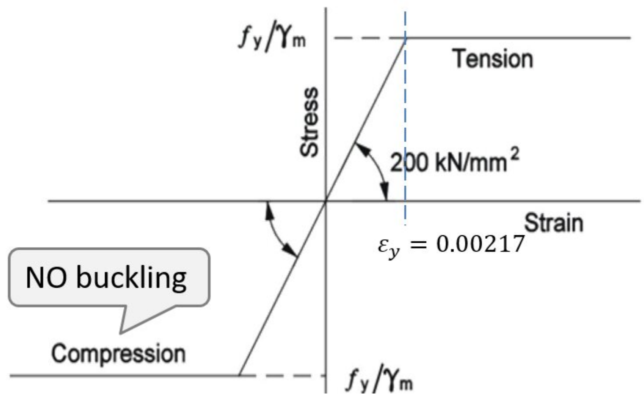

Figure 2.3: Elastoplastic stress-strain curve for steel

2.2.1 Steel (Tension and Compression)

If buckling of steel (in compression) is prevented by design (the steel bars are embeded in concrete), the behavior of steel is identical in tension and compression HKCC2013 [7]. **Idealized Curve**: The shape is elastic-plastic and symmetric in tension and compression.

The **maximum stress** in steel is given by:

$$f_{s,max} = \frac{f_y}{\gamma_m} = \frac{f_y}{1.15} = 0.87 f_y$$

where:

- f_y: Characteristic yield strength, e.g., $f_y = 500\,\mathrm{N/mm^2}$.
- $\gamma_m = 1.15$: Material safety factor.
- According to the adopted idelized curve the maximum stress in steel is the yield stress: $f_y/\gamma_m = 0.87 f_y$ ($\gamma_m = 1.15$).
- the stress in steel depends on the strain, and is given by a different function before and after yielding:

$$f_s(\varepsilon_s) = \begin{cases} E_s \varepsilon_s & \text{if } \varepsilon_s < \varepsilon_y \text{ (linear elastic branch – Hooke's law)} \\ 0.87 f_y & \text{if } \varepsilon_s \geq \varepsilon_y \text{ (plastic branch – maximum stress)} \end{cases} \quad (2.1)$$

where:

- $\varepsilon_y = 0.00217$ is the **yield strain**, and
- $E_s = 200\,\mathrm{kN/mm^2} = 200\,\mathrm{GPa}$ is the **Young's modulus**.

Nomenclature:

f_y : Characteristic tensile strength of steel according to HK2003

f_{yk} : Characteristic tensile strength of steel according to EC2

2.2.2 Concrete

Idealized Curve: The tensile resistance of concrete is ignored, and only compressive behavior is considered. The shape is parabolic-plastic: parabolic up to max stress, then plastic (constant stress) until ε_{cu}. This is a simplification since the actual distribution of concrete stresses (from experimental data) usually exhibits a softening branch.

The **maximum stress** in concrete is given by:

$$f_{c,max} = 0.67\frac{f_{cu}}{\gamma_m} = 0.45 f_{cu}$$

where

- f_{cu}: Characteristic compressive strength of concrete.
- $\gamma_m = 1.5$: Material safety factor of compression (Table 1.5).
- 0.67: Empirical factor from tests (not a safety factor).

The **maximum strain** of concrete at ULS is given by:

$$\varepsilon_{cu} = \begin{cases} 0.0035 = 0.35\,\%, & \text{for } f_{cu} \leq 60\,\mathrm{N/mm^2} \\ 0.0035 - 0.00006(f_{cu} - 60), & \text{for } f_{cu} > 60\,\mathrm{N/mm^2} \end{cases} \tag{2.2}$$

The HKConcrete2004 idealized stress-strain curve differs compared with the BS 8110 design curve in that the values of E_c are generally smaller than those of BS 8110.

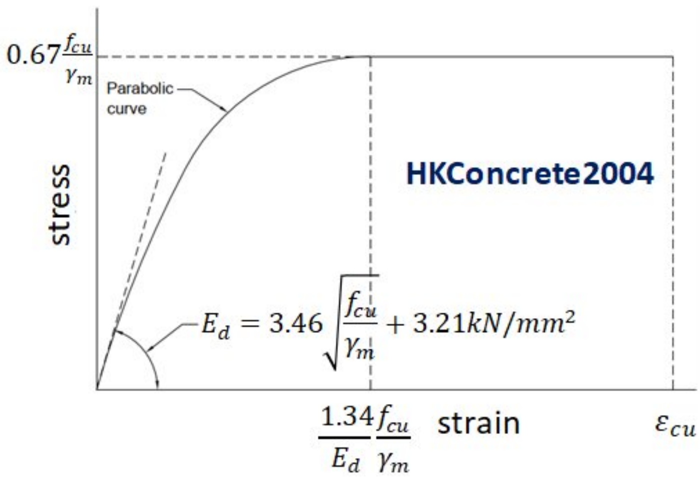

Figure 2.4: Parabolic-plastic stress-strain curve for concrete

Additionally, when $f_{cu} > 60\,\mathrm{N/mm^2}$ (high-strength concrete), $\varepsilon_{cu} < 0.0035$.

Nomenclature:

f_{cu} : Characteristic compressive strength of concrete according to HK2003

f_{ck} : Characteristic compressive strength of concrete according to EC2

2.2.3 Concrete stress: Equivalent Rectangular Stress Block

The parabolic-plastic stress distribution of concrete at ULS creates challenges when calculating the resultant force and its point of application (see Figure 2.5 left). To simplify design calculations several methods have been proposed. HKCC2013 [7], EC2 [11] and ACI-318 [1] all make use of the concept of an equivalent rectangular stress block, which was pioneered by Whitney. BS 8110 [13] employs a rectangular-parabolic stress block.

The HKCC2013 code (Clause 6.1.2.4) replaces the nonlinear (parabolic-plastic) stress distribution of concrete at failure with a rectangular stress block of dimensions (Figure 2.5 right) $0.45 f_{cu} \times s$, i.e., of uniform stress $0.45 f_{cu}$ and of compression zone depth $s = s(f_{cu})$:

$$s = s(f_{cu}) = \begin{cases} s = 0.9x & \text{for} \qquad f_{cu} \leq 45\,\mathrm{N/mm^2} \\ s = 0.8x & \text{for } 45\,\mathrm{N/mm^2} < f_{cu} \leq 70\,\mathrm{N/mm^2} \\ s = 0.72x & \text{for } 70\,\mathrm{N/mm^2} < f_{cu} \leq 100\,\mathrm{N/mm^2} \end{cases} \tag{2.3}$$

This simplifies the problem drastically, as now the resultant compressive force of concrete is a simple function of the compression zone depth:

$$F_{cc} = 0.45 f_{cu} \cdot bs(f_{cu}) \tag{2.4}$$

with its point of application at the middle of the stress block depth, i.e., at distance $s/2$ from the edge of the stress block.

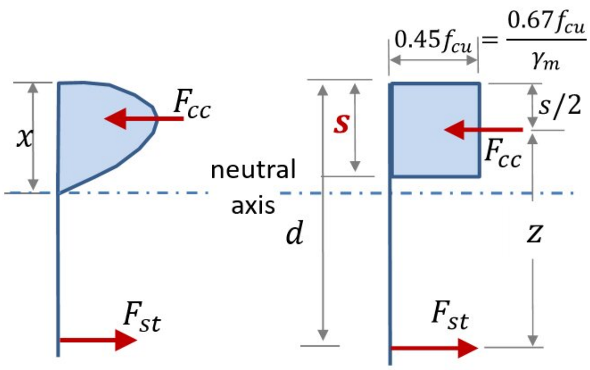

Figure 2.5: The nonlinear distribution of concrete stresses (left) is replaced by a simple stress-block (right).

2.3 Design of Singly Reinforced Cross Sections

We are ready to solve the design problem of the cross section analytically and establish a design formulae. To this end, we adopt the three basic assumptions (Section 2.1.1) and calculate the involved compressive force based on the simplified rectangular stress block for concrete (Section 2.2.3).

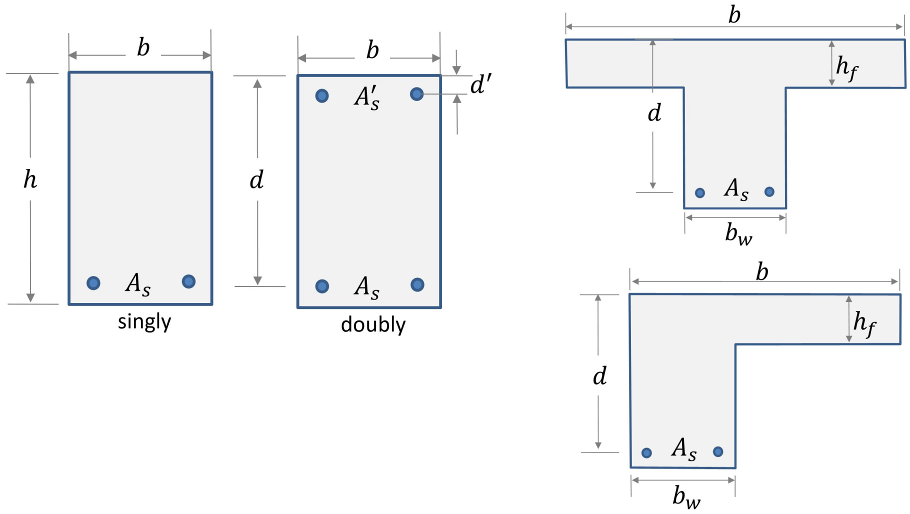

Figure 2.6: Common cross section geometries: (left) rectangular section with tension steel only, (middle) rectangular section with both tension and compression steel, (right) flanged sections of T or L shapes.

2.3.1 Problem Formulation

To start form the easiest possible design case, we consider a rectangular cross-section, which represents the simplest geometry (see Figure 2.6). and examine the case of pure bending under a moment M, with no axial force applied ($N = 0$).

Referring to Figure 2.7, we analyze a bending moment M that induces tension on the bottom side of the cross-section (i.e., a positive or sagging bending moment). Note that the depiction of the external moment M in Figure 2.6 is inaccurate, as it suggests a moment within the plane of the cross-section. This is misleading, as the moment is applied perpendicular to the plane of the cross-section. Figure 1.4 (bottom right) illustrates the correct axis of the applied moment. For convenience however, we adopt this visual representation of the external moment (and later axial force) throughout these notes, despite its inaccuracy.

Additionally, we assume that the section does not require compression steel reinforcement ($A_s' = 0$), meaning the cross-section is treated as singly reinforced. This assumption will be verified based on the design outcome.

Design Problem

The objective of design is to determine the area of tension steel reinforcement A_s required to resist the moment at the ultimate limit state (ULS) of failure. The strain and stress profiles are described as follows:

- **Stress Block**: The parabolic-plastic stress distribution at ULS is simplified into a rectangular shape (see Section 2.2.3),

- **Strain Profile**: The strain varies linearly, with maximum compressive strain equal to ε_{cu} at the top of the section for a sagging moment, and the tensinle strain ε_s at the steel level (see Section 2.1.1).

Given:

- Geometry: rectangular cross section with width b, height h, and effective depth d (distance from top to centroid of tensile steel)
- Material strengths: compressive strength of concrete f_{cu}, tensile strength of steel f_y
- Material strains: Ultimate strain of concrete ε_{cu}, and strain in tension steel ε_s
- Design bending moment M as calculated from structural analysis.

Unknowns:

1. **Reinforcement**: Required tensile steel area A_s.
2. **One of the three parameters**: x, s, z. Any of these three geometric parameters: the depth of the compression zone s, the neutral axis depth x, the lever arm of the two forces z, suffices to determine the other two. Since s and x are related $s = s(x)$ see Table 2.3 and simple geometry dictates that $z = d - s\frac{f_{cu}}{2}$. Thus, any of the three parameters could serve as the second unknown.

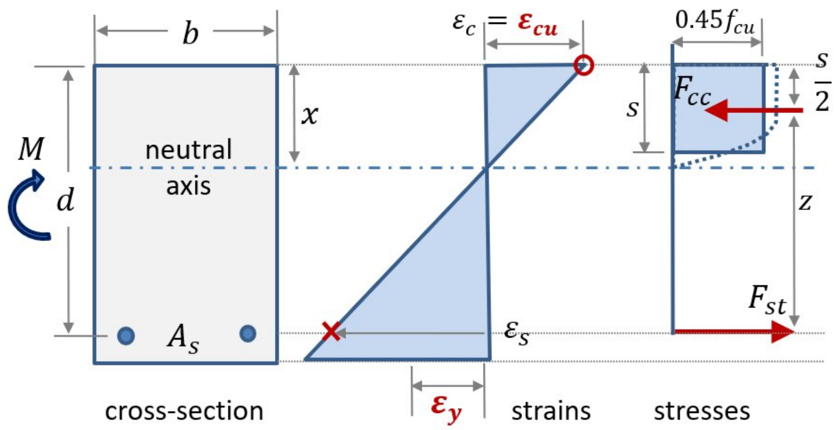

Figure 2.7: Singly reinforced rectangular section in bending

2.3.2 Design Formulae

Forces

There are only two forces applied on the cross-section of Figure 2.7: the compressive force of concrete F_{cc} and the tensile force of steel F_{st}.

- **Tensile Steel Force (F_{st})** is calculated as 'stress' times 'area': $F_{st} = f_s \cdot A_s$.
  - A_s is the required steel reinforcement area.
  - f_s is the stress of steel:

$$f_s = \begin{cases} E_s\varepsilon_s & \text{if } \varepsilon_s < \varepsilon_y \quad \text{linear elastic branch - Hooke's law} \\ 0.87f_y & \text{if } \varepsilon_s \geq \varepsilon_y \quad \text{plastic branch - maximum stress} \end{cases}$$

- **Concrete Compressive Force (F_{cc})** equals 'stress' times 'area': $F_{cc} = 0.45 f_{cu} \cdot (b \cdot s)$. From the simplified rectangular stress block (HKCC2013 code), it follows that:
  - The concrete compressive stress is $= 0.45 f_{cu}$.
  - Compression zone depth s (depends on f_{cu} (Equation 2.3))
  - Compression zone area = $b \cdot s$.

Equilibrium Conditions
The two forces applied on the cross-section (F_{cc} and F_{st}) are both along the horizontal axis (Figure 2.7). Therefore there are only two equilibrium conditions:

- **Force Equilibrium along x-axis**:

$$\sum F_x = 0 \rightarrow F_{cc} = F_{st} \rightarrow 0.45 f_{cu} s b = 0.87 f_y A_s \tag{2.5}$$

- **Moment Equilibrium**:

$$\sum M = 0 \rightarrow M = F_{cc} \cdot z, \text{ where } z = d - \frac{s}{2} \rightarrow$$
$$M = F_{cc} z = 0.45 f_{cu} s b (d - s/2) \tag{2.6}$$

Mathematical solution
We have formulated the design problem of the singly reinforced rectangular cross section into a mathematical problem of **two equations**, the force (Eq 2.5) and moment (Eq 2.6) equilibrium, involving **two unknowns**. The unknowns are the required reinforcement steel area A_s, and any one of the following parameters: the depth of the compression zone s, or the neutral axis depth x, or the lever arm of the two forces z. In the following we choose the lever arm z as the second unknown, and proceed to solve the two equations analytically for the two unknowns A_s and z.

Analytical derivations

- Concrete compressive force: $F_{cc} = 0.45 f_{cu} \cdot bs$
- Steel tensile force: $F_{st} = 0.87 f_y A_s$
- Lever arm: $z = d - s/2 \rightarrow s = 2(d - z)$
- The moment equilibrium yields a quadratic equation:

$$M = F_{cc} z = 0.45 f_{cu} bs \cdot z = 0.45 f_{cu} b 2(d-z) \cdot z \rightarrow M - 0.9 f_{cu} bdz + 0.9 f_{cu} bz^2 = 0 \rightarrow$$
$$\left(\frac{z}{d}\right)^2 - \frac{z}{d} + \frac{M}{0.9 bd^2 f_{cu}} = 0 \rightarrow \left(\frac{z}{d}\right)^2 - \frac{z}{d} + \frac{K}{0.9} = 0, \text{ where } K = \frac{M}{bd^2 f_{cu}}$$

- Solving the quadratic eqn gives the lever arm z : $\frac{z}{d} = 0.5 + \sqrt{0.25 - \frac{K}{0.9}}$
- Substituting the lever arm z in the moment equilibrium $M = F_{st} z$ yields the steel area: $A_s = M/(0.87 f_y z)$

2.3.3 Solution Procedure:

In summary the **solutions steps** are:

1. Calculate the bending moment ratio.

$$K = \frac{M}{bd^2 f_{cu}} \tag{2.7}$$

2. Solve the quadratic equation to determine z (Table 2.1):

$$\frac{z}{d} = 0.5 + \sqrt{0.25 - \frac{K}{0.9}} \tag{2.8}$$

3. Compute the required area of (tension) steel:

$$A_s = \frac{M}{0.87 f_y z} \tag{2.9}$$

4. Choose reinforcement steel bars that cover the A_s requirement (Table 2.2).

5. Check the reinforcement ratio (for reasons we will explain in Section 2.3.7):

$$\rho_{\text{min}} \leq A_s/(bh) \leq \rho_{\text{max}}$$

Table 2.1: Values of z/d for Different K Values

K	0.05	0.06	0.07	0.09	0.10	0.12	0.14	0.156
z/d	0.941	0.928	0.915	0.887	0.873	0.842	0.807	0.775

Table 2.2: Area of Reinforcement (mm^2) based on Diameter and Number of bars

Diameter	Number of bars							
(mm)	1	2	3	4	5	6	7	8
6	28	56	84	113	141	169	197	226
8	50	100	150	201	251	301	351	402
10	78	157	235	314	392	471	549	628
12	113	226	339	452	565	678	791	904
16	201	402	603	804	1005	1206	1407	1608
20	314	628	942	1256	1570	1884	2199	2513
25	490	981	1472	1963	2454	2945	3436	3927
32	804	1608	2412	3216	4021	4825	5629	6433
40	1256	2513	3769	5026	6283	7539	8796	10053

2.3.4 Limits and Conditions

Recall that a basic assumption of our design formulae is that the section is singly reinforced, or in simple words, that it needs only tension steel reinforcement and no compression steel reinforcement. For reasons we will explain in Section 2.4 this is true when the bending moment ratio K is smaller than a limit value $K \leq K'$.

- If $K \leq K'$, the section requires only tension steel and can therefore be designed as a singly reinforced section.

- If $K > K'$, the section requires both compression and tension steel, necessitating design as a doubly reinforced section.
- The depth to the neutral axis x is constrained by $x_{\text{min}} \leq x \leq x_{\text{max}}$. Due to the simple geometry, these limits on x correspond to equivalent limits on the lever arm z, given by $z_{\text{min}} \leq z \leq z_{\text{max}}$, as x and z are inversely related.
- with the aid of x_{max} the lower limit of lever arm z [clause 6.1.2.4(b)] is defined:

$$z_{\text{min}} = z_{\text{min}}(f_{cu}) \rightarrow z = d - \frac{s(x)}{2} \rightarrow z_{\text{min}} = d - \frac{s(x_{\text{max}}(f_{cu}))}{2}$$

- If the lever arm $z > z_{\text{max}}$, the design calculations proceed by setting $z = z_{\text{max}}$. This typically occurs for small moments relative to the cross-section size.
- If $z < z_{\text{min}}$, this implies $x > x_{\text{max}}$, since z and x are inversely related, and $z = z_{\text{min}}$ corresponds to $x = x_{\text{max}}$. In this case, the section should either be designed as a doubly reinforced section by adding compression steel and setting $x = x_{\text{max}}$, or it indicates an error in the design.

Effect of Concrete Grade f_{cu}
These limit values of K', x, and z, all depend on the concrete grade f_{cu} (and the **moment redistribution** to be discussed in a later Section 2.5). Here is how:

- $\varepsilon_{cu} = \varepsilon_{cu}(f_{cu})$ according to Equation 2.2
- $s = s(f_{cu})$ according to Equation 2.3
- $x_{\text{max}} = x_{\text{max}}(f_{cu})$ [see Table 2.3]
- $x_{min} = 0.11d$; $z_{max} = 0.95d$
- substituting $z = z_{\text{min}}$ in the quadratic Equation 2.8 and solving for K returns $K = K_{\text{max}} = K'(f_{cu})$:

$$K' = 0.225 - 0.9\left(\frac{z_{\text{min}}(f_{cu})}{d} - 0.5\right)^2 \tag{2.10}$$

The effect of f_{cu} on K', x_{max}, s, and z_{min} is tabulated in Table 2.3.

$f_{cu} \leq \mathrm{N/mm^2}$	$\frac{x_{\text{max}}}{d}$	$\frac{s}{x}$	$\frac{z_{\text{min}}}{d}$	K'
$45\,\mathrm{N/mm^2}$	0.5	0.9	0.775	0.157
$70\,\mathrm{N/mm^2}$	0.4	0.8	0.840	0.121
$100\,\mathrm{N/mm^2}$	0.33	0.72	0.881	0.094

Table 2.3: The effect of f_{cu} on x_{max}, s, z_{min}, and K'.

2.3.5 Example 1: Design of a singly reinforced rectangular section

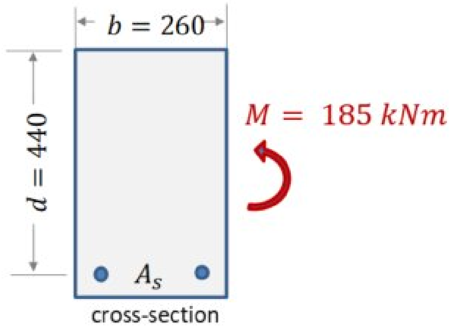

Figure 2.8: Singly reinforced rectangular section in bending

Design a singly reinforced R/C rectangular section given that:

- the design moment to be resisted by the section is $M = 185\,\mathrm{kN\,m}$,
- the dimensions of the cross-section are $b = 260\,\mathrm{mm}$ and $d = 440\,\mathrm{mm}$,
- the characteristic material strengths are $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.

Solution steps

1. Calculate the (dimensionless) bending moment ratio K and compare with K':

$$K = \frac{M}{bd^2 f_{cu}} = \frac{185 \cdot 10^6}{260 \cdot 440^2 \cdot 30} = 0.122 < 0.157 = K'$$

   $K < K' \rightarrow$ singly reinforced section.

2. Calculate the lever arm (Equation 2.8):

$$\frac{z}{d} = \left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) = \left(0.5 + \sqrt{0.25 - \frac{0.122}{0.9}}\right) = 0.838 \rightarrow z = 369\,\mathrm{mm}$$

   Check $z_{\text{min}}/d = 0.775 \leq z/d = 0.838 \leq 0.95 = z_{\text{max}}/d \rightarrow$ OK

3. Required tension steel area (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{185 \cdot 10^6}{0.87 \cdot 500 \cdot 369} = 1153\,\mathrm{mm}^2$$

4. Choose bars: 4T20: 4 high strength steel bar of $20\,\mathrm{mm}$ diameter ($A_s = 1256\,\mathrm{mm}^2$).

5. Check the steel ratio (Figure 2.16):

$$\rho_{min} = 0.3\,\% < \rho = \frac{A_s}{b \cdot h} = \frac{1256}{260 \cdot 490} = 1.0\,\% < \rho_{max} = 2.5\,\% \rightarrow \text{OK.}$$

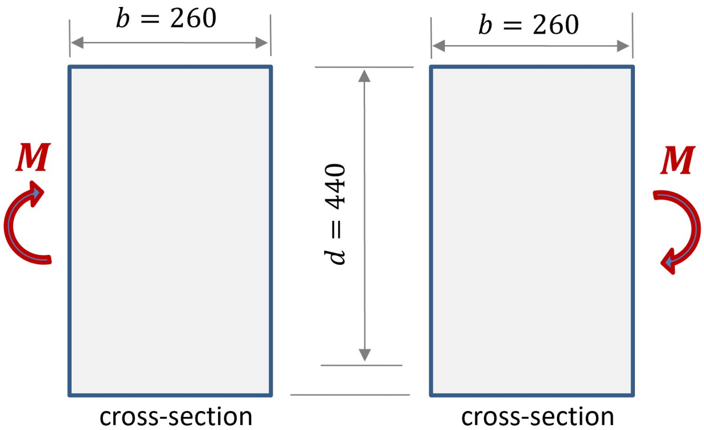

Figure 2.9: Singly reinforced rectangular section: positive vs negative moment.

Example 1: continued

Consider again the design **example but this time assume the design bending moment is negative**. How would the design change?
Answer

- Since the section is rectangular and hence symmetric to positive and negative moments, all calculations remain the same. Thus, we will provide again 4T20.
- The vital difference is we will provide them on the top of the section where is now the tension side.

2.3.6 Example 2: Design of a singly reinforced rectangular section

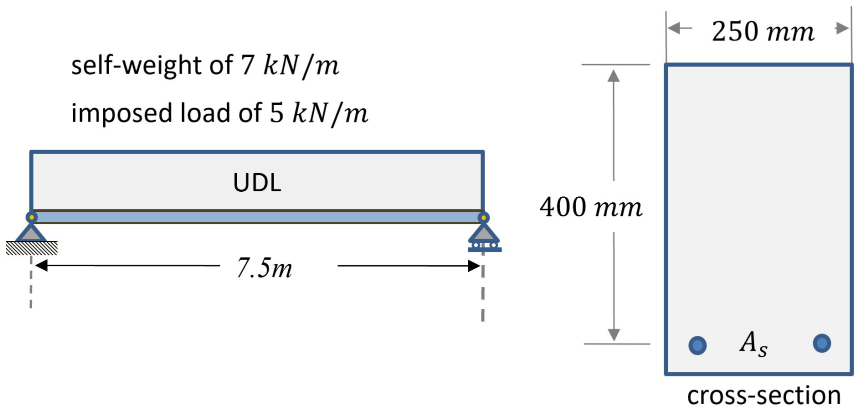

Figure 2.10: Simply supported beam with rectangular section.

Design a singly reinforced R/C rectangular section given that:

- the dimensions of the cross-section are $b = 250\,\mathrm{mm}$ and $d = 400\,\mathrm{mm}$,
- the characteristic material strengths are $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.

- the dead load = $7\,\mathrm{kN/m}$ and the live load = $5\,\mathrm{kN/m}$.

Solution
The solution process is very similar with that of the previous example. Unlike the previous example, in this one we are not given the design moment, but we have to calculate it from the design load combination.

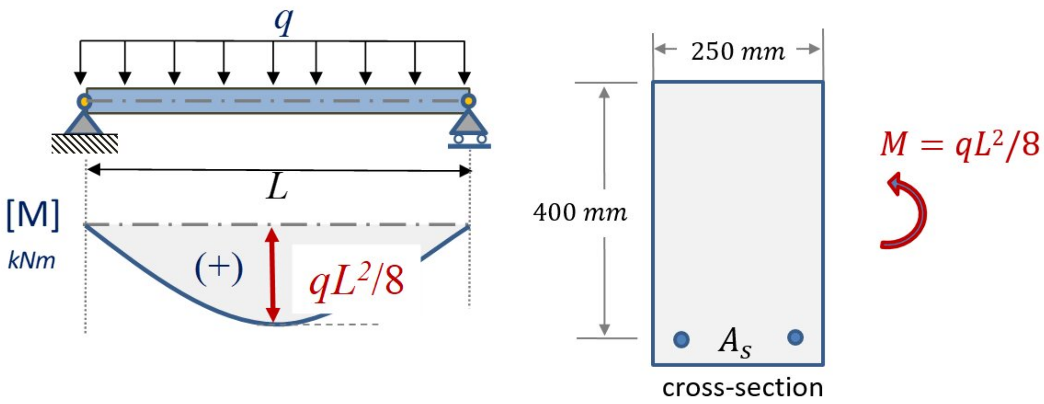

Figure 2.11: Bending moment diagram of a simply supported beam under UDL

- **Design load** : $q = 1.4 \cdot 7 + 1.6 \cdot 5 = 17.8\,\mathrm{kN/m}$;
- **Design moment**:$M = \frac{q \cdot L^2}{8} = \frac{17.8 \cdot 7.5^2}{8} = 125.2\,\mathrm{kN\,m}$.

From this point on, the design solution is identical with the previous problem (2.3.6).
Solution steps

1. Calculate the (dimensionless) bending moment ratio K and compare with K':

$$K = \frac{M}{bd^2 f_{cu}} = \frac{125.2 \cdot 10^6}{250 \cdot 400^2 \cdot 30} = 0.104 < 0.157 = K'$$

   $K < K' \rightarrow$ singly reinforced section.

2. Calculate the lever arm (Equation 2.8):

$$\frac{z}{d} = \left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) = \left(0.5 + \sqrt{0.25 - \frac{0.104}{0.9}}\right) = 0.867 \rightarrow z = 347\,\mathrm{mm}$$

   Check $z_{\text{min}}/d = 0.775 \leq z/d = 0.867 \leq 0.95 = z_{\text{max}}/d \rightarrow$ OK

3. Required tension steel area (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{125.2 \cdot 10^6}{0.87 \cdot 500 \cdot 347} = 829\,\mathrm{mm}^2$$

4. Choose bars: 3T20: 3 high strength steel bars of $20\,\mathrm{mm}$ diameter ($A_s = 942\,\mathrm{mm}^2$).

5. Check the **steel ratio** (Figure 2.16):

$$\rho_{min} = 0.3\,\% < \rho = \frac{A_s}{b \cdot h} = \frac{942}{250 \cdot 450} = 0.84\,\% < \rho_{max} = 2.5\,\% \rightarrow \text{OK}$$

2.3.7 Types of failure and Reinforcement ratio

A fundamental goal of engineering design is to achieve the desired failure mode. In structural engineering the strong preference is on prioritizing 'ductile' failure modes that only occur after sufficient 'warning'. Simply put, in a reinforced concrete beam there are two components that can fail, either concrete under crushing or steel after first yielding and then fracturing after deforming excessively in plastic manner.

Type 1: Balanced Section

In a balanced section, tension steel yielding ($\varepsilon_s = \varepsilon_y$) and concrete crushing ($\varepsilon_c = \varepsilon_{cu}$) occur at failure simultaneously. Recall that from the linear strain distribution in the section (Figure 2.12) it follows that:

$$\varepsilon_s = \varepsilon_{cu}\frac{d-x}{x}$$

The depth of the neutral axis x is then:

$$x = \frac{d}{1+\frac{\varepsilon_s}{\varepsilon_{cu}}}$$

For ($f_{cu} \leq 45\,\mathrm{N/mm^2}$) the ultimate strain of concrete is $\varepsilon_{cu} = 0.0035$, while for high-yield steel the yield strain of steel is $\varepsilon_y = 0.00217$. Hence the neutral axis depth x_{bal} is (Figure 2.12):

$$x_{bal} = \frac{d}{1+0.00217/0.0035} = 0.617d \rightarrow x_{bal} = 0.617d \tag{2.11}$$

Therefore, to ensure yielding of the tension steel at the Ultimate Limit State (ULS) the depth of the neutral axis x must be less than x_{bal}:

$$x < 0.617d$$

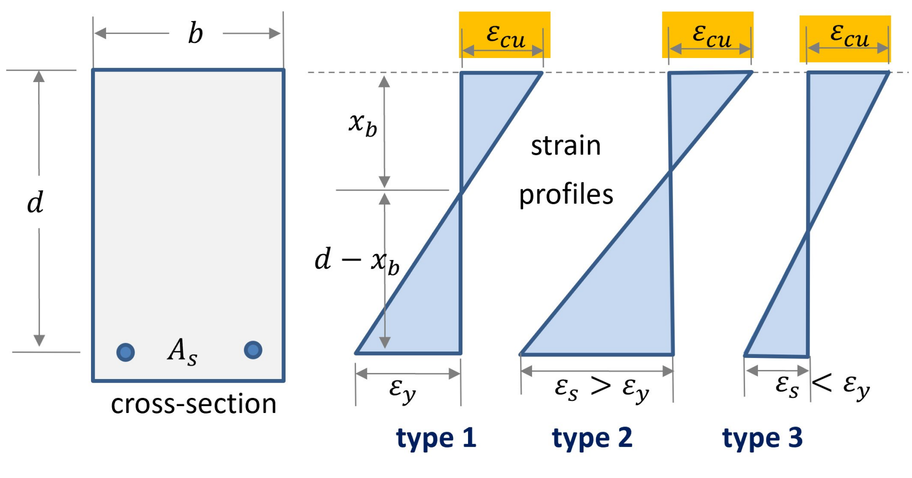

Figure 2.12: Strain profiles of the three different failure modes

Type 2: Under-Reinforced Section

In an under-reinforced section, the steel yields $\varepsilon_s > \varepsilon_y$ before the concrete strain reaches $\varepsilon_c < \varepsilon_{cu}$. The failure of an under-reinforced beam is characterized by large steel strains (Figure 2.13), leading to extensive cracking of the concrete and substantial deflection. The

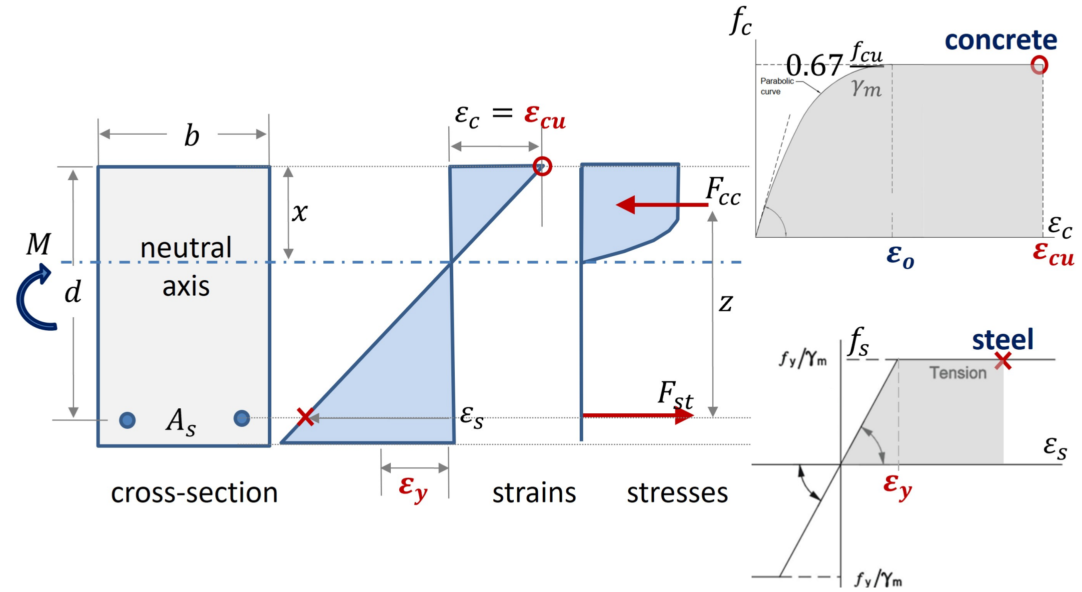

Figure 2.13: Under-reinforced beam at ULS failure: Overview of strain and stress profiles, and resulting forces (left). Stress-strain diagrams for concrete and steel (right).

ductility of such a beam provides ample warning of impending failure. For this reason, design aims for under-reinforcement. However, it is not allowed to design a beam without any steel reinforcement or with a very low steel ratio. Therefore, the reinforcement ratio should be controlled by:

$$\rho > \rho_{\text{min}} \tag{2.12}$$

Type 3: Over-Reinforced Section

In an over-reinforced section, the concrete strain reaches its ultimate value ε_{cu} before the steel strain attains the yield strain ε_y. Consequently, failure in an over-reinforced beam is initiated by the crushing of the concrete, while the steel strain remains relatively low, leading to a brittle failure mode (see Figure 2.14). This type of failure is characterized by minimal deflection and limited cracking in the tension zone. Due to its sudden nature, such failure occurs with little warning and can often be explosive.

Figure 2.14: Brittle failure of an over-reinforced beam - concrete crushing.

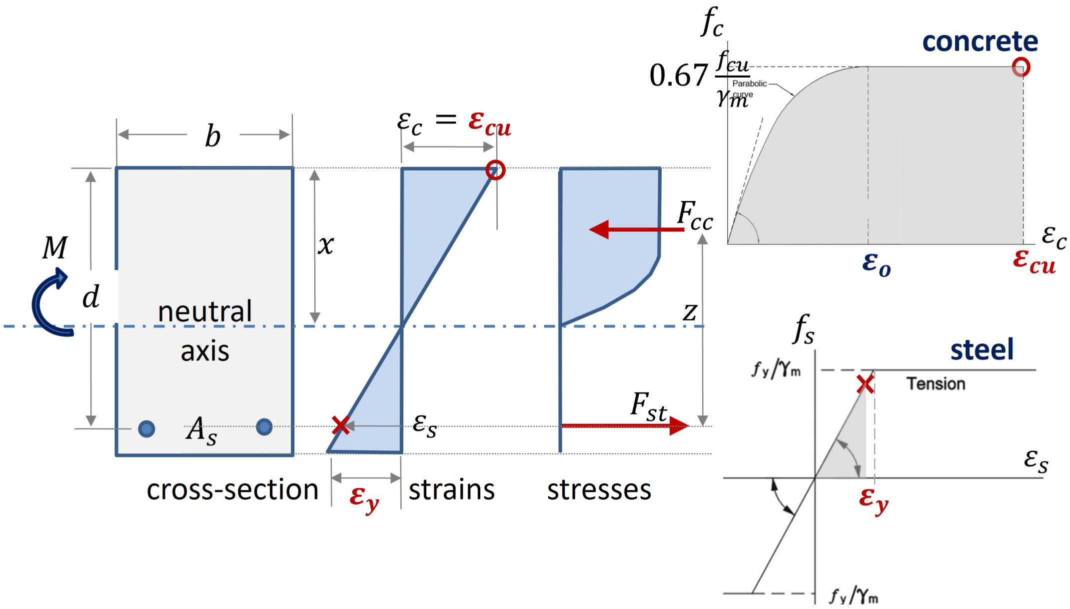

Figure 2.15: Over-reinforced beam at ULS failure: Overview of strain and stress profiles, and resulting forces (left). Stress-strain diagrams for concrete and steel (right).

In summary,

- **Type 1: Balanced Section** limit case is defined when at the ULS of failure: $\varepsilon_c = \varepsilon_{cu}$ and $\varepsilon_s = \varepsilon_y$ occurs when $x = x_{bal}$
- **Type 2: Under-Reinforced Section** preferred ductile failure mode: $\varepsilon_c = \varepsilon_{cu}$ and $\varepsilon_s > \varepsilon_y$ occurs when at the ULS $x < x_{bal}$.
- **Type 3: Over-Reinforced Section** undesired brittle failure to be avoided : $\varepsilon_c = \varepsilon_{cu}$ and $\varepsilon_s < \varepsilon_y$ occurs when at the ULS $x > x_{bal}$.

Reinforcement Ratio

The reinforcement ratio is defined as the area of steel reinforcement (provided in the cross-section) divided by the cross-sectional area of the concrete:

$$\rho = \frac{\text{area of reinforcement provided}}{\text{gross cross-sectional area of the concrete}} = \frac{A_s}{bh} \tag{2.13}$$

According to HKCC2013, the reinforcement ratio limits are as follows (Figure 2.16):

- Without ductility demands
  - $\rho_{\max} = 4\,\%$ (clause 9.2.1.3)
  - $\rho_{\min} = 0.13\,\%$ (for high-yield steel) (clause 9.2.1.1)
- With ductility demands
  - $\rho_{\max} = 2.5\,\%$ (clause 9.9.1.1)
  - $\rho_{\min} = 0.3\,\%$ (clause 9.9.1.1)

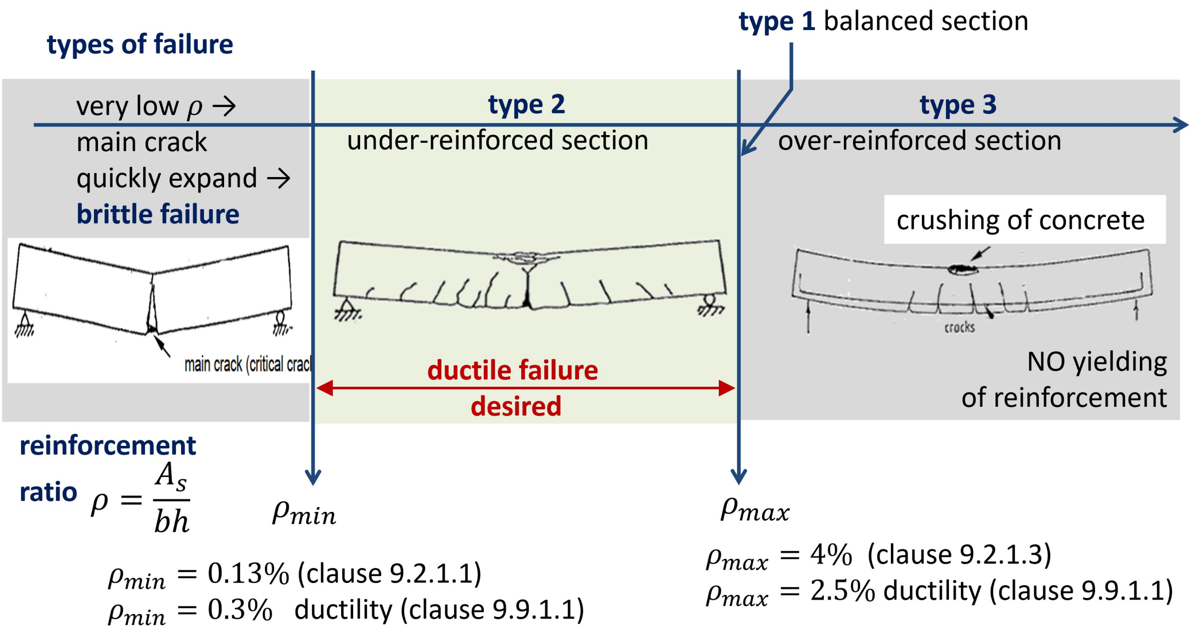

Figure 2.16: Types of failure and reinforcement ratio limits.

Ductility is required in most structural elements such as beams and columns. So The preferred limits are usually $\rho_{\text{max}} = 2.5\,\%$ and $\rho_{\text{min}} = 0.3\,\%$. An exception is slabs where ductility is not required, and hence the limits are $\rho_{\text{max}} = 4\,\%$ and $\rho_{\text{min}} = 0.13\,\%$.

Final remarks

- The goal of the cross section design is an under-reinforced section so that steel yields before concrete crushes, providing warning via crack formation and crack propagation before collapse.
- The scope of the maximum (ρ_{max}) limit is to avoid over-reinforcement, and ensure steel yields before concrete crushes ('under-reinforced' section). results in an over-reinforced section brittle failure.
- If $\rho > \rho_{\text{max}}$ we need to increase the size of the section (b or d) to reduce ρ.
- The scope of the minimum limit ρ_{min} is to resist cracking from minor stresses (e.g., self-weight, temperature) and prevent brittle failure from insufficient tension capacity post-cracking.
- If $\rho < \rho_{\text{min}}$ we need to provide steel reinforcement area that covers the ρ_{min} demand.
- Preferred reinforcement ratio value in most cases for beams is $\rho \approx 1\,\%$.

2.3.8 Capacity Checking of a Singly Reinforced Section

We refer again to the mechanical system of Figure 2.7 but under different assumptions and goal. Specifically, here the goal is to calculate the bending moment resistance M_R of a designed (and hence known) cross section. This need emerges often when checking the results of a design, or when we need to verify the capacity of a design modification. In such a case, the following quantities are given:
Given:

- Geometry: rectangular cross section with width b, height h, and effective depth d (distance from top to centroid of tensile steel)
- Material strengths: compressive strength of concrete f_{cu}, tensile strength of steel f_y
- Reinforcement: tension steel area A_s.

Unknowns:

1. bending moment M_R resistance.
2. Neutral Axis (NA) depth x (or equivalently the depth of the compression zone s).

Design Formulae

Forces There are only two forces applied on the cross-section of (Figure 2.7): the compressive force of concrete F_{cc} and the tensile force of steel F_{st}.

- **Tensile Steel Force (F_{st})**:
  - Force equals 'stress' times 'area': $F_{st} = f_s \cdot A_s$.
  - the stress in steel depends on the strain, and is given by a different function before and after yielding:

$$f_s = \begin{cases} E_s \varepsilon_s & \text{if } \varepsilon_s < \varepsilon_y \quad \text{linear elastic branch - Hooke's law} \\ 0.87 f_y & \text{if } \varepsilon_s \geq \varepsilon_y \quad \text{plastic branch - maximum stress} \end{cases}$$

- **Concrete Compressive Force (F_{cc})**:
  Form the simplified rectangular stress block (HKCC2013 code), it follows that:
  - Compression zone depth s (depends on f_{cu} Equation 2.3)
  - Forces equals 'stress' times 'area': $F_{cc} = 0.45 f_{cu} \cdot (b \cdot s)$.

Equilibrium Conditions The two forces applied on the cross-section (F_{cc} and F_{st}) are both along the horizontal axis Figure 2.7. Therefore there are only two equilibrium conditions:

- **Force Equilibrium along x-axis**:

$$\sum F_x = 0 \rightarrow F_{cc} = F_{st} \rightarrow 0.45 f_{cu} b_s = 0.87 f_y A_s$$

- **Moment Equilibrium**:

$$\sum M = 0 \rightarrow M_R = F_{st} z = 0.87 f_y A_s \left(d - \frac{s}{2}\right)$$

We have formulated the capacity problem of the singly reinforced rectangular cross section as a mathematical problem of **two equations** (the force and moment equilibrium) involving **two unknowns**. The unknowns are the bending moment M_R resistance, and the depth of the compression zone s.

Solution

- From the Force Equilibrium it follows: $s = \frac{0.87 f_y A_s}{0.45 f_{cu} b}$
- From the Moment Equilibrium: $M_R = 0.87 f_y A_s \left(d - \frac{0.87 f_y A_s}{0.9 f_{cu} b} \right)$

2.3.9 Example: Capacity checking of a singly reinforced rectangular section

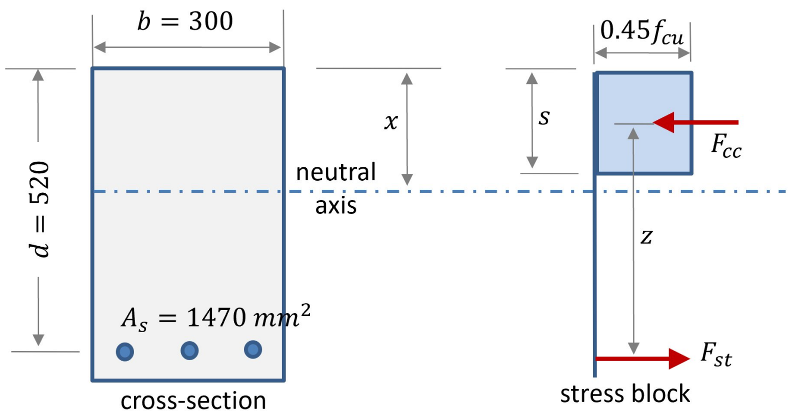

Figure 2.17: Singly reinforced rectangular section - capacity checking

Determine the ultimate moment of resistance M_R of a singly reinforced R/C rectangular cross-section with:

- dimensions of the cross-section are $b = 300\,\mathrm{mm}$ and $d = 520\,\mathrm{mm}$,
- characteristic material strengths are $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.
- area of tension steel $A_s = 1470\,\mathrm{mm}^2$

Solution steps

1. The force equilibrium returns the compression zone depth s:

$$F_{cc} = F_{st} \rightarrow 0.45 f_{cu} bs = 0.87 f_y A_s \rightarrow$$

$$0.45 \cdot 30 \cdot 300 \cdot s = 0.87 \cdot 500 \cdot 1470 \rightarrow s = 158\,\mathrm{mm}$$

2. The neutral axis depth is:

$$x = \frac{s}{0.90} = \frac{158}{0.90} = 176\,\mathrm{mm}$$

Since

$$\varepsilon_s = \varepsilon_{cu} \frac{d - x}{x} = 0.0035 \cdot \frac{520 - 176}{176} = 0.0068 > \varepsilon_y$$

yielding is confirmed.

3. The moment equilibrium returns the ultimate moment of resistance of the cross-section at the ULS:

$$M_R = 0.87 f_y A_s \left(d - \frac{s}{2}\right) = 0.87 \cdot 500 \cdot 1470 \left(520 - \frac{158}{2}\right) \cdot 10^{-6} = 282.0\,\mathrm{kN\,m}$$

2.4 Design of Doubly Reinforced Cross Sections

Consider the behavior of a singly reinforced cross-section at the ultimate limit state (ULS) under an increasing external bending moment M (Figure 2.7). As M increases, the bending moment ratio K rises, the neutral axis depth ratio x/d (see Equation 2.8) increases nonlinearly (see Figure 2.18), and the lever arm ratio z/d (see Equation 2.8) decreases nonlinearly (see Figure 2.18). Recall that z and x are inversely related.

When $K = K'$, the neutral axis depth reaches its maximum allowable value, $x = x_{\text{max}}$. Beyond this point, further increases in the neutral axis depth are not permitted, preventing any additional increase in the concrete compression force and, consequently, the moment resistance.

To enhance the moment resistance beyond this limit, compression steel must be introduced, requiring the design of a doubly reinforced section.

In summary:

- If $K < K'$, the section is designed as a singly reinforced beam.
- If $K > K'$, the section must be designed as a doubly reinforced beam.

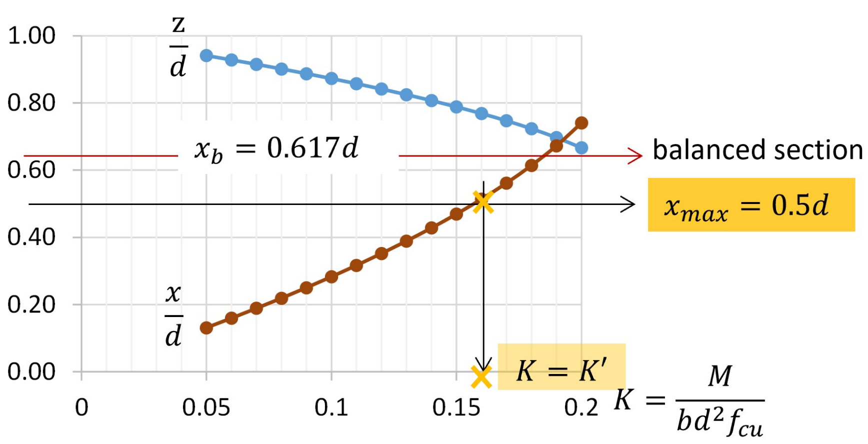

Figure 2.18: Types of failure and reinforcement ratio limits.

2.4.1 Problem Formulation

Let us formulate the design problem of a doubly reinforced section mathematically. From condition $K > K'$, we infer that the neutral axis has attained its maximum value $x = x_{max}$, or equivalently that the lever arm its minimum value $z = z_{min}$. Our objective is to determine the required tension and compression steel.

Given:

- Geometry: rectangular cross section with width b, height h, effective depth d, and d' equal with the distance from top to centroid of compression steel.
- Material strengths: compressive strength of concrete f_{cu}, tensile strength of steel f_y.
- Design bending moment M.

Unknowns:

1. Reinforcement: Required tension steel area A_s.
2. Reinforcement: Required compression steel area A_s'.

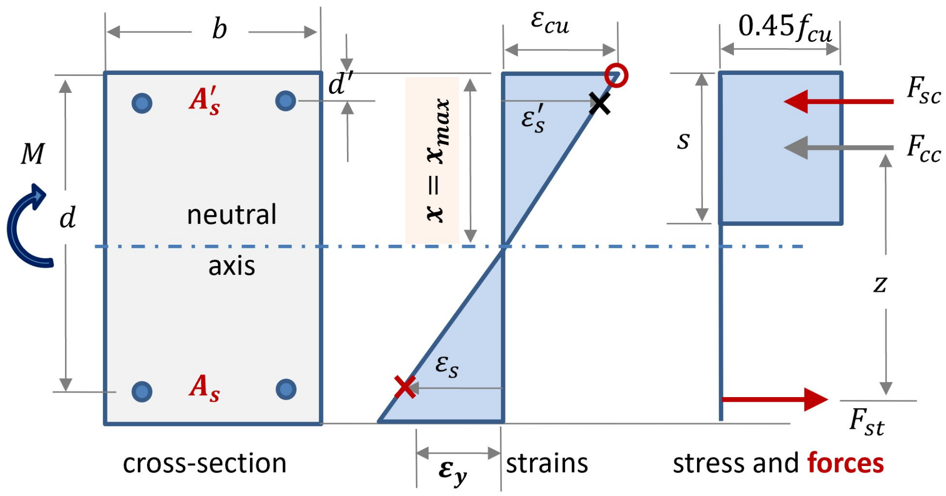

Figure 2.19: Analysis of a doubly reinforced section.

2.4.2 Design Formulae

Forces: There are now three forces applied on the cross-section of (Figure 2.19):

- **Tensile Steel Force (F_{st}):**
  - $F_{st} = 0.87 f_y A_s$ (tension steel, assumed yielded).
- **Compressive Steel Force (F_{sc}):**
  - $F_{sc} = f_{sc} A_s'$, where the compressive stress is $f_s'(x) = \begin{cases} E_s \varepsilon_s' & \text{if } \varepsilon_s' < \varepsilon_y \\ 0.87 f_y & \text{if } \varepsilon_s' \geq \varepsilon_y \end{cases}$
  - where the strain of the compression steel is equal with $\varepsilon_s' = \varepsilon_{cu} \frac{x-d'}{x}$.
- **Concrete Compressive Force (F_{cc}):**
  Adopting again the simplified rectangular stress block (HKCC2013 code), the comrpessive force of concrete

$$F_{cc} = 0.45 f_{cu} (b \cdot s)$$

  - From $x = x_{max} \rightarrow$ it follows that the compression zone depth has is also at its maximum $s = s_{max}$. Therefore, F_{cc} is known and at its maximum value:

$$F_{cc} = 0.45 f_{cu} b \cdot s_{max}$$

  - Recall that both s_{max} and x_{max} limits depend on f_{cu} (Table 2.3). For $f_{cu} \leq 45\,\mathrm{MPa} \rightarrow s_{max} = 0.9 x_{max}$ (Equation 2.3):

$$F_{cc} = 0.45 f_{cu} b (0.9 \cdot 0.5d) = 0.2025 f_{cu} bd$$

Equilibrium Conditions: The equilibrium conditions are still two:

- **Force Equilibrium along x-axis**:

$$\sum F_x = 0 \rightarrow F_{cc} + F_{sc} = F_{st} \rightarrow 0.45 f_{cu} b s_{max} + f_s'(x) A_s' = 0.87 f_y A_s$$

- **Moment Equilibrium** about the centroid of NA:

$$\sum M_{NA} = 0 \rightarrow M = F_{cc}\left(x - \frac{s}{2}\right) + F_{sc}(x - d')y + F_{st}(d - x)$$
$$\sum M_{NA} = 0 \rightarrow M = 0.45 f_{cu} bs \left(x - \frac{s}{2}\right) + f_s'(x) A_s'(x - d') + 0.87 f_y A_s (d - x)$$

- or **Moment Equilibrium** about any other point.

We have formulated the design problem of a doubly reinforced rectangular cross section as a mathematical problem of **two equations** (one force and one moment equilibrium) involving **two unknowns**. Unlike the case of singly reinforced sections $K \leq K'$ where the two unknowns of design are the tension steel area A_s, and neutral axis depth x (or equivalently the lever arm z), in the doubly reinforced section case $K > K'$ the two unknowns are the tension steel area A_s, and the compression steel area A_s'. The neutral axis depth x is fixed at its maximum value x_{max} (or equivalently the lever arm z at its minimum value $z = z_{min}$) and is no longer an unknown since for $K > K'$ it follows that $x = x_{max}$.

- The moment equilibrium about the centroid A_s is:

$$M = 0.156 f_{cu} bd^2 + 0.877 f_y A_s'(d - d')$$

- Hence, the required **compression steel** area A_s' is:

$$A_s' = \frac{(K - K') f_{cu} bd^2}{0.87 f_y (d - d')} \tag{2.14}$$

- The force equilibrium returns the required **tension steel** area A_s:

$$A_s = \frac{K' f_{cu} bd^2}{0.877 f_y z} + A_s' \tag{2.15}$$

where: $K = M/(bd^2 f_{cu})$ and K' is given from Table 2.3 when no redistribution of moments is accounted for $\leq 10\%$, or from Table 2.6 when redistribution of moments taken into account.

2.4.3 Example 1: Design (No Redistribution)

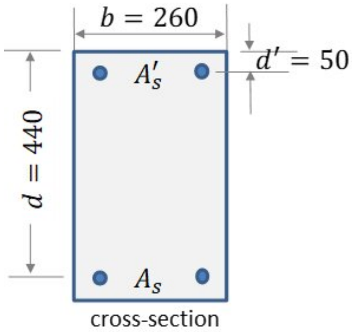

Figure 2.20: Doubly reinforced rectangular section in bending

Design of a doubly reinforced rectangular section with moment redistribution applied not greater than $10\,\%$.

Given:

- Geometry $b = 260\,\mathrm{mm}$, $d = 440\,\mathrm{mm}$, $d' = 50\,\mathrm{mm}$.
- Characteristic material strengths $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.
- Design moment $M = 270\,\mathrm{kN\,m}$, take $\beta_b \geq 0.9$ (no moment redistribution).

Solution steps

1. Check the (dimensionless) bending moment ratio:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{270 \cdot 10^6}{260 \cdot 440^2 \cdot 30} = 0.179 > K' = 0.156 \rightarrow$$

   compression steel required, we need to design a doubly reinforced section.

2. NA depth: $x = x_{max} = 0.5d = 220\,\mathrm{mm}$, $s = s_{max} = 0.9x = 198\,\mathrm{mm}$.

3. Compression steel required, assuming yielding: $f_{sc} = 0.87 f_y$, is (Equation 2.14):

$$A_s' = \frac{(K - K')bd^2 f_{cu}}{0.87 f_y (d - d')} = \frac{(0.179 - 0.156) \cdot 260 \cdot 440^2 \cdot 30}{0.87 \cdot 500 \cdot (440 - 50)} = 205\,\mathrm{mm}^2.$$

   Strain of compression steel confirms yielding $\varepsilon_s' > \varepsilon_y$:

$$\varepsilon_s' = \varepsilon_{cu} \frac{x - d'}{x} = 0.0035 \frac{220 - 50}{220} = 0.0027 > 0.00217 = \varepsilon_y$$

4. Tension steel required (Equation 2.15):

$$A_s = \frac{K' bd^2 f_{cu}}{0.87 f_y z} + A_s' = \frac{0.156 \cdot 30 \cdot 260 \cdot 440^2}{0.87 \cdot 500 \cdot (0.775 \cdot 440)} + 205 = 1793\,\mathrm{mm}^2$$

   Provide: $A_s' = 2T12$ bars ($226\,\mathrm{mm}^2$), $A_s = 4T25$ bars ($1963\,\mathrm{mm}^2$).

5. Check the **steel ratio** (Figure 2.16):

$$\rho_{min} = 0.3\,\% < \rho = \frac{A_s}{b \cdot h} = \frac{1963}{260 \cdot 490} = 1.5\,\% < \rho_{max} = 2.5\,\% \rightarrow \text{OK.}$$

   For bending tension steel A_s, and not A_s', should be used in the calculation of ρ.

2.4.4 Example: Design with Moment Redistribution

Design a section with a moment redistribution of $20\,\%$ ($\beta_b = 0.8$).

Given :

- Geometry $b = 260\,\mathrm{mm}$, $d = 440\,\mathrm{mm}$, $d' = 50\,\mathrm{mm}$.
- Characteristic material strengths $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.
- Design moment = $228\,\mathrm{kN\,m}$.

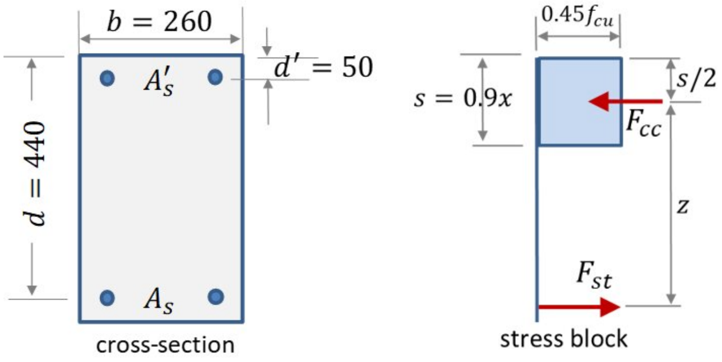

Figure 2.21: .

Solution steps:

1. Calculate the lever arm:
   - Redistribution of moments is $20\,\% \rightarrow \beta_b = 0.8$.
   - depth of the neutral axis: $x = (\beta_b - 0.4)d = (0.8 - 0.4)d = 0.4d$
   - compression zone depth: $s = 0.9x = 0.36d$,
   - lever arm: $z = d - \frac{s}{2} = 0.82d$.
2. Moment of resistance of the concrete section:

$$M_{cc} = F_{cc}z = 0.45 f_{cu} b s z =$$

$$= 0.45 \cdot 30 \cdot 260 \cdot 0.36 \cdot 440 \cdot 10^{-3} \cdot 0.82 \cdot 440 \cdot 10^{-3} = 201\,\mathrm{kN\,m}$$

   Since $M_{cc} = 201\,\mathrm{kN\,m} < 228\,\mathrm{kN\,m}$ **compression steel is required**.

3. Check if compression steel has yielded: $\frac{50}{176} = 0.28 < 0.38$, confirming that the compression steel has yielded.
4. Required compression steel A_s' is (Equation 2.14):

$$A_s' = \frac{M - M_{cc}}{0.87 f_y (d - d')} = \frac{(228 - 201) \cdot 10^6}{0.87 \cdot 500 \cdot (440 - 50)} = 159\,\mathrm{mm}^2$$

5. Required tension steel area A_s is (Equation 2.15):

$$A_s = \frac{M_{cc}}{0.87 f_y z} + A_s' = \frac{201 \cdot 10^6}{0.87 \cdot 500 \cdot 0.82 \cdot 440} + 159 = 1281 + 159 = 1440\,\mathrm{mm}^2$$

   Provide: 2T25 + 2T20 ($A_s = 1609\,\mathrm{mm}^2$).

6. Check the **steel ratio** (Figure 2.16):

Alternative solution using the design equations of the HKCC2013 code:

1. **(K'):** Note that because of moment redistribution $20\,\%$ ($\beta_b = 0.8$) $K' = 0.132$ and not 0.156 (see Table 2.6)

2. Calculate the dimensionless bending moment ratio K and compare with K':

$$K = \frac{M}{bd^2 f_{cu}} = \frac{228 \cdot 10^6}{260 \cdot 440^2 \cdot 30} = 0.151 > K' = 0.132 \rightarrow$$

   $K > K' \rightarrow$ compression steel required, doubly reinforced section.

3. Calculate the lever arm ($z = z_{min}$) (or take it from Table 2.6):

$$z = z_{min} = d\left(0.5 + \sqrt{0.25 - \frac{K'}{0.9}}\right) = d\left(0.5 + \sqrt{0.25 - \frac{0.132}{0.9}}\right) = 0.82d$$

4. Calculate the required compression steel A'_s is (Equation 2.14):

$$A'_s = \frac{(K - K')f_{cu}bd^2}{0.87f_y(d - d')} = \frac{(0.151 - 0.132) \cdot 30 \cdot 260 \cdot 440^2}{0.87 \cdot 500 \cdot (440 - 50)} = 169\,\mathrm{mm}^2$$

5. Calculate the required tension steel A_s is (Equation 2.15):

$$A_s = \frac{K'bd^2 f_{cu}}{0.87f_y z} + \left(\frac{f'_s}{0.87f_y}\right)A'_s = \frac{0.132 \cdot 30 \cdot 260 \cdot 440^2}{0.87 \cdot 500 \cdot (0.82 \cdot 440)} + 169 = 1439\,\mathrm{mm}^2$$

   Provide: 2T25 + 2T20 ($A_s = 1609\,\mathrm{mm}^2$).

6. Check the **steel ratio** (Figure 2.16):

$$\rho_{max} = 0.3\,\% < \rho = \frac{A_s}{b \cdot h} = \frac{1609}{260 \cdot 490} = 1.3\,\% < \rho_{max} = 2.5\,\% \rightarrow \text{OK}$$

   For bending A_s, and not A'_s, should be used in the calculation of ρ.

2.4.5 Capacity Checking of a Doubly Reinforced Section

We refer again to the mechanical system of a doubly reinforced section (Figure 2.19) but this time under different assumptions and objective. Specifically, the objective here is to calculate the bending moment resistance M_R of a designed (and hence known) cross section can provide. In such a case the following quantities are given:

Given:

- Geometry: rectangular cross section with width b, height h, effective depth d (distance from top to centroid of tensile steel), d' equal with the distance from top to centroid of compression steel.
- Material strengths: compressive strength of concrete f_{cu}, tensile strength of steel f_y
- Reinforcement: tension steel area A_s and compression steel area A_s'.

Unknowns:

1. Bending moment M_R resistance.
2. Neutral Axis (NA) depth x (or equivalently s).

Design Formulae

Forces There are three forces applied on the cross-section of (Figure 2.19) and two equilibrium conditions.

- Tensile Steel Force (F_{st}) assuming tension steel has yielded:

$$F_{st} = 0.87 f_y A_s$$

- Compressive Steel Force (F_{sc}):

$$F_{sc} = f_{sc} A_s', \text{ where the compressive steel stress is } f_s'(x) = \begin{cases} E_s \varepsilon_s' & \text{if } \varepsilon_s' < \varepsilon_y \\ 0.87 f_y & \text{if } \varepsilon_s' \geq \varepsilon_y \end{cases}$$

  where the strain of the compression steel is equal with $\varepsilon_s' = \varepsilon_{cu} \frac{x-d'}{x}$. Note that $\varepsilon_s' \geq \varepsilon_y$ corresponds to: $x > 2.63d'$ (for $\varepsilon_{cu} = 0.0035$).
- Concrete Compressive Force (F_{cc}): According to the simplified rectangular stress block (HKCC2013 code), the compressive force of concrete is:

$$F_{cc} = 0.45 f_{cu} \cdot (b \cdot s)$$

Equilibrium Conditions The two equilibrium conditions are:

- Force Equilibrium along x-axis:

$$\sum F_x = 0 \rightarrow F_{cc} + F_{sc} = F_{st} \rightarrow \sum F_x = 0 \rightarrow 0.87 f_y A_s = 0.45 f_{cu} bs + f_s'(x) A_s'$$

- Moment Equilibrium about the centroid of A_s':

$$\sum M_{As'} = 0 \rightarrow M_R = 0.45 f_{cu} bs \left(d - \frac{s}{2}\right) + f_s'(x) A_s' (d - d')$$

We have formulated the capacity problem of the doubly reinforced rectangular cross section as a mathematical problem of **two equations** (the force and moment equilibrium) involving **two unknowns**. The unknowns are the bending moment M_R resistance, and the depth of the compression zone s.

Solution

- From the Force Equilibrium it follows:

$$s = \frac{0.87 f_y A_s - f_s'(x) A_s'}{0.45 f_{cu} b}$$

- From the Moment Equilibrium:

$$M_R = 0.45 f_{cu} b s \left(d - \frac{s}{2} \right) + f_s'(x) A_s' (d - d')$$

2.4.6 Example: Checking of a Doubly R/C Rectangular Section

Check the capacity of a rectangular doubly reinforced concrete section.

Given:

- Geometry $b = 280\,\mathrm{mm}$, $d = 510\,\mathrm{mm}$, $d' = 50\,\mathrm{mm}$.
- Characteristic material strengths $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.
- Reinforcement: tension steel area $A_s = 2410\,\mathrm{mm}^2$.
- Reinforcement: compression steel area $A_s' = 628\,\mathrm{mm}^2$.

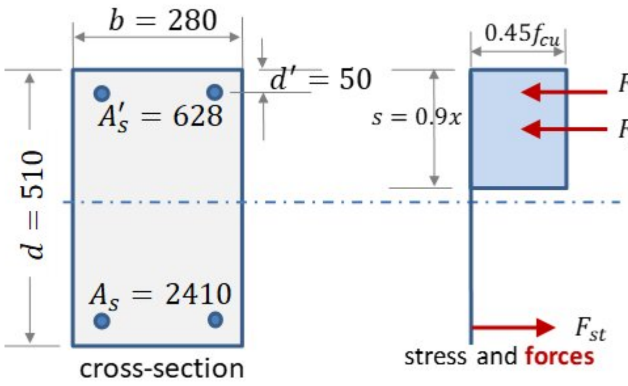

Figure 2.22

Solution steps:

1. Assume both tension and compression steel have yielded at failure:

$$F_{st} = F_{cc} + F_{sc} \rightarrow 0.87 f_y A_s = 0.45 f_{cu} bs + 0.87 f_y A_s'$$

2. Calculate the depth of the stress block:

$$s = \frac{0.87 f_y (A_s - A_s')}{0.45 f_{cu} b} = \frac{0.87 \cdot 500(2410 - 628)}{0.45 \cdot 30 \cdot 280} = 205\,\mathrm{mm}$$

3. Verify the tension steel has yielded:

$$x = \frac{s}{0.9} = \frac{205}{0.9} = 228\,\mathrm{mm} < x_{bal} = 0.617d = 315\,\mathrm{mm}$$

   Hence, the tension steel has yielded, as assumed.

4. Verify the compression steel has yielded:

$$\frac{d'}{x} = \frac{50}{228} = 0.22 < 0.38$$

   Hence, the compression steel has yielded, as assumed.

5. Calculate the ultimate moment by taking moments about the tension steel:

$$M_R = 0.45 f_{cu} bs \left(d - \frac{s}{2}\right) + 0.87 f_y A_s'(d - d') =$$

$$= 0.45 \cdot 30 \cdot 280 \cdot 205 \left(510 - \frac{205}{2}\right) + 0.87 \cdot 500 \cdot 628(510 - 50) = 441.4\,\mathrm{kN\,m}$$

2.4.7 Example 2.4.6 continued

Discussion:

When $\frac{d'}{x} > 0.38$, the compression steel will not have yielded; thus

$$F_{cc} + F_{sc} = F_{st} \rightarrow 0.405 f_{cu} bx + \left(E_s \varepsilon_{cu} \frac{x - d'}{x} \right) A'_s = 0.87 f_y A_s$$

Solution steps:

1. Equilibrium condition:

$$0.405 f_{cu} bx^2 + \left(E_s \varepsilon_{cu} A'_s - 0.87 f_y A_s \right) x - E_s \varepsilon_{cu} d' A'_s = 0$$

   where $f'_s = (E_s \varepsilon'_s) = \left(E_s \varepsilon_{cu} \frac{x-d'}{x} \right)$.

2. Then the moment of resistance:

$$M = F_{cc} \left(d - 0.9x/2 \right) + F_{sc} \left(d - d' \right)$$

3. Solving the equation yields the value of x.

4. Taking moments about the tension reinforcement, the moment of resistance of the cross-section is given by:

$$F_{cc} = 0.45 f_{cu} b (0.9x)$$

   and

$$F_{sc} = f'_s A'_s = \left(E_s \varepsilon'_s \right) A'_s = \left(E_s \varepsilon_{cu} \frac{x - d'}{x} \right) A'_s$$

2.5 Moment Redistribution

2.5.1 Motivational Example

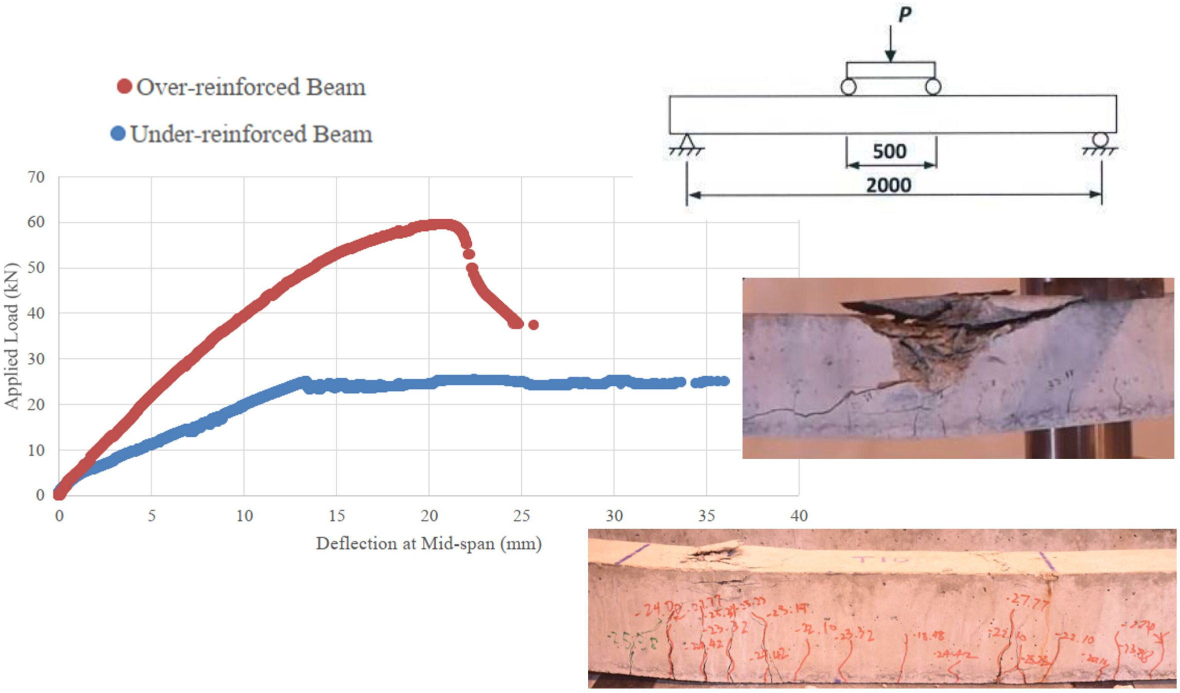

Figure 2.23: Red curve: Brittle failure of an over-reinforced beam (top right) - concrete crushing. Blue curve: Ductile failure of an under-reinforced beam (bottom right)- steel yielding.

Figure 2.23 showcases the load-deflection behavior of two beams tested in the Structural Engineering Lab of the HKUST as part of this course assignment. It highlights a clear contrast. The blue curve represents a (properly designed) under-reinforced beam, while the red curve shows an over-reinforced beam. The under-reinforced beam exhibits significant ductility, achieving a displacement ductility factor of about $\mu \approx 36/12 = 3$. As the load increases, cracks form and spread in its tension zone, enabling significant plastic rotation and giving a clear warning of failure. In contrast, the over-reinforced beam offers greater strength and stiffness but fails brittlely right after peak strength. Specifically, the over-reinforced beam fails due to concrete crushing in the compression zone, which occurs after almost no warning and produces minimal ductility. This contradiction emphasizes a key lesson in structural design, the interplay of: stiffness, strength, and ductility.

The under-reinforced beam, though less stiff and strong, constitutes better design. It signals failure through visible cracking and deformation, providing ample warning of the pending collapse. From the purpose of design, the question is how to ensure our beam (structural member) or more generally our structure (structural system) displays ample ductility?

2.5.2 Plastic Hinge Behavior and Moment Redistribution

Ductility at the structural system level is achieved through the formation of **plastic hinges** at critical sections of the structure (e.g., beam supports or mid-spans). Figure 2.24 illustrates an idealized elastic-plastic material. This is the simplest modeling of the behavior

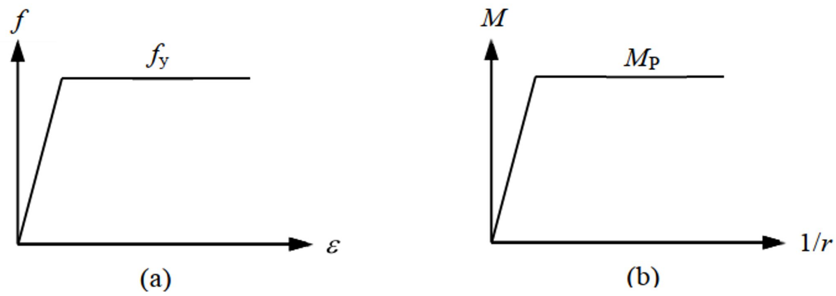

Figure 2.24: Plastic hinge behavior.

of a R/C concrete cross section before and after yielding, in terms of stress-strain (Figure 2.24 left) and Moment - rotation (Figure 2.24 right). For a beam section subjected to an increasing moment M, the curvature $1/r$ increases linearly with M until the plastic moment of resistance M_P is reached (Figure 2.24(b)). Beyond this point, the curvature increases indefinitely as the section behaves as a plastic hinge (see the stress-strain behavior in Figure 2.24(a)). The formation of plastic hinges requires large rotations which can only be achieved by yielding and plastic deformation of the tension steel. To ensure sufficient strains in the tension reinforcement, the neutral axis depth x is restricted by:

$$x \leq (\beta_b - 0.4)d \leq 0.5d,$$

where the **moment redistribution ratio** β_b is defined as:

$$\beta_b = \frac{\text{moment after redistribution}}{\text{moment before redistribution}} \leq 1,$$

or equivalently,

$$\beta_b = \frac{\text{design moment}}{\text{elastic moment}} \leq 1.$$

Key observations:

- If the hinge rotation does not cause concrete crushing, additional hinges may form at other sections until a mechanism develops. The formation of additional plastic hinges capable of large plastic deformations is what generates ductility on the level of the structural system.
- Plastic hinge behavior is limited by **concrete failure**, which restricts rotation due to the small compressive strain capacity of concrete.

2.5.3 Elastic vs. Elastic-Plastic Analysis

This section compares **linear elastic** and **elastic-plastic** analyses in the context of moment distribution and collapse load, highlighting the role of moment redistribution.

Key Differences

- **Linear Elastic Analysis**: Assumes no yielding, resulting in overestimated support moments and a linear stress-strain relationship throughout the structure.
- **Elastic-Plastic Analysis**: Accounts for the formation of plastic hinges at critical sections (e.g., supports or mid-spans), enabling moment redistribution and revealing a higher collapse load than elastic analysis.
- **Collapse Mechanism**: Occurs when sufficient plastic hinges form to create a mechanism (e.g., three hinges in a fixed-fixed-end beam).

Moment Redistribution

Moment redistribution occurs in elastic-plastic analysis as plastic hinges form, reducing support moments compared to elastic predictions. This behavior is quantified by the **moment redistribution ratio** (β_b), defined as the ratio of the actual (plastic) moment to the elastic moment. The maximum neutral axis depth ($x_{\max}$) decreases with β_b, reflecting increased redistribution. Structural design incorporating moment redistribution is termed **plastic design**.

Practical Implications

- Elastic-plastic analysis demonstrates that beams can sustain higher loads before collapse by redistributing moments from heavily loaded sections to underutilized ones.
- In practical design, β_b is used to adjust neutral axis depth limits (x) rather than requiring full plastic analysis, simplifying the process while accounting for ductility.
- The concept is illustrated in continuous beams, where elastic-plastic behavior reduces peak moments at supports, enhancing overall structural efficiency.

We will explore the effects of elastic-plastic behavior on moment distribution through examples of continuous beams. By calculating the collapse load and β_b, we will quantify the extent of moment redistribution and validate the increased capacity compared to elastic predictions.

2.5.4 Example 1: Fixed-Fixed Beam

Consider a fixed-fixed beam of length L, subjected to a uniform load w, with ultimate moment capacity M_p at all sections. Assume the cross section of the beam behaves in the elastic plastic manner of Figure 2.24 for both positive and negative moments. Thus, all sections in the beam yield at a maximum moment M_p . We will analyze the behavior of the beam from zero until collapse load and we will calculate the collapse load and the moment redistribution ratio.

- **Elastic range**: For low load levels $0 \leq w \leq w_y$, the behavior of the beam is linear elastic. Consequently, the structure is described by an indeterminate fixed-fixed beam and the bending moments at its supports and mid-span are respectively: $M_{\text{support}} = wL^2/12$ and $M_{\text{mid}} = wL^2/24$.
- The first two plastic hinges form at the supports when the moment at the supports reaches the ultimate moment capacity of the cross-section $M = M_p$. Let the corresponding load level to be w_y. It holds:

  $$M_p = \frac{w_y L^2}{12}$$

  The moment at the midspan for the w_y load is $M_B = w_y L^2/24$.
- **Post-elastic range**: For higher load levels w_a with $w = w_y + w_a > w_y$ the structural system effectively changes since both supports behave as (plastic) hinges. Therefore, the analysis of the same structure for loads higher than w_y assumes a simply supported beam system. Accordingly, the additional moment developed at the mid-span is the elastic moment (from the fixed-fixed beam system) and the post-elastic moment (from

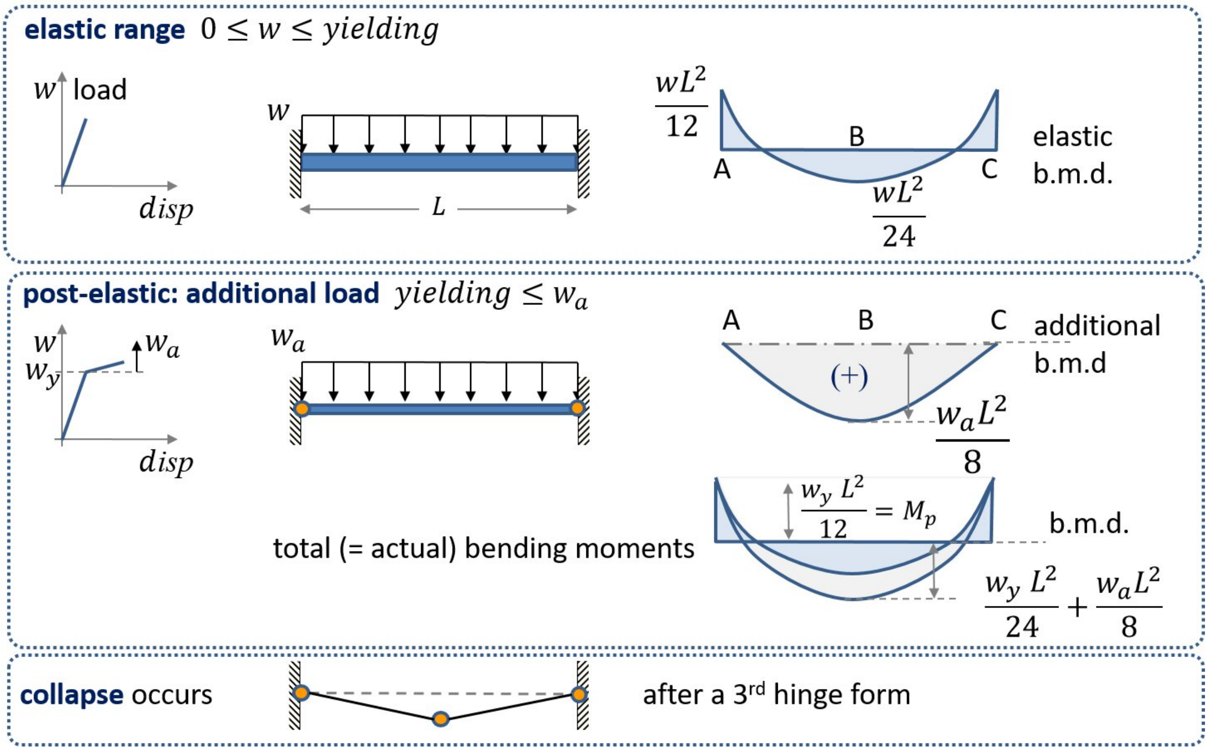

Figure 2.25: Structural system in the elastic range of behavior (top), post-elast (middle) and collapse mechanism (bottom).

the simply supported system) and increases until the ultimate moment capacity of the cross-section M_p.

$$\frac{w_y L^2}{24} + \frac{w_a L^2}{8} = M_p \rightarrow \frac{w_y L^2}{24} + \frac{w_a L^2}{8} = \frac{w_y L^2}{12} \rightarrow w_a = \frac{w_y}{3}$$

Once a third plastic hinge form, this time in the midspan, the structure becomes a mechanism, it deforms without further increase of the load until collapse. The load level that generates the third hinge is the collapse load w_u, since with the third hinge the structure turns to a mechanism, and the collapse mechanism is generated (Figure 2.25).

- **Collapse load**: Therefore, the collapse load is $33\,\%$ higher than the yield load:

$$w_u = w_y + w_a = w_y + \frac{w_y}{3} = 1.33 w_y$$

- **Redistribution ratio at supports**: For the maximum load the structure can handle w_u, the moment of the supports accounting for yielding is $w_y L^2/12$, whereas ignoring yielding of the cross section the moment at C is $w_u L^2/12$. Therefore, the moment redistribution factor is:

$$\beta_b = \frac{w_y L^2/12}{w_u L^2/12} = \frac{w_y}{w_u} = \frac{w_y}{1.33 w_y} = 0.75 \le 1$$

In other words, the elastic analysis overestimates support moments compared to a more realistic plastic analysis.

2.5.5 Example 2: Two-Span Continuous Beam with Point Loads

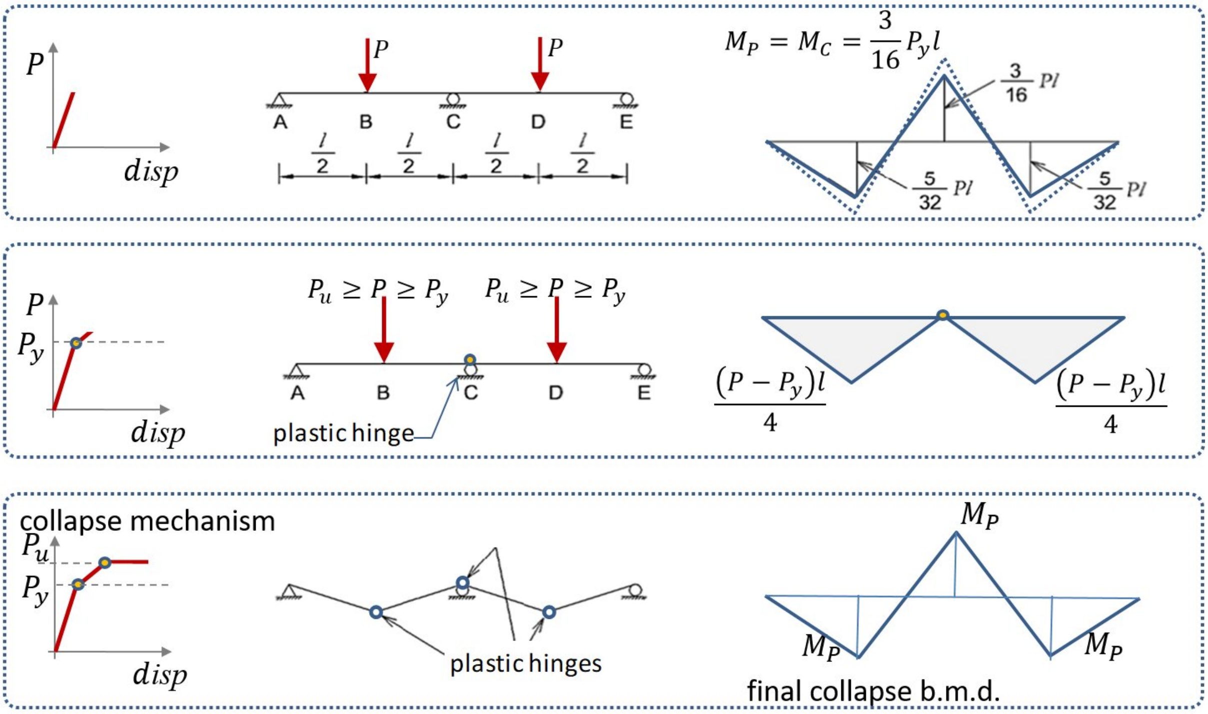

Figure 2.26: Structural system in the elastic range of behavior (top), post-elast (middle) and collapse mechanism (bottom).

Consider a two-span continuous beam (spans L, supports A, B, C), with point loads P at the midspan of each span, and ultimate moment capacity M_p. Assume the cross section of the beam behaves in the elastic plastic manner of Figure 2.24 for both positive and negative moments. We will again analyze the structural behavior of the beam from zero until collapse load and we will calculate the collapse load and the moment redistribution factor.

- **Elastic range**: For low load levels $0 \leq P \leq P_y$, the behavior of the beam is linear elastic. Consequently, the structure is described by a 2-span continuous beam and the bending moments at the support and mid-spans are respectively: $M_C = \frac{3Pl}{16}$ and $M_{mid} = \frac{5Pl}{32}$.

- The first plastic hinge forms at the support C when the moment reaches the ultimate moment capacity of the cross-section $M = M_p$. Let the corresponding load level to be P_y. It holds:

$$M_p = \frac{3P_y l}{16} \rightarrow \quad P_y = \frac{16M_p}{3l}$$

  The moment at the midspans for P_y is $M_B = M_D = 5P_y l/32$.

- **Post-elastic range**: For higher load levels $P = P_y + P_a > P_y$ the structural system effectively changes to two simply supported beams as the support at C behaves as a (plastic) hinge. For additional load P_a, the additional midspan moment is:

$$\Delta M_{\text{midspan}} = \frac{P_a l}{4}$$

  The load level that generates the third hinge is the collapse load w_u, since with the third hinge the structure turns to a mechanism.

- **Collapse load**: The collapse mechanism is generated (Figure 2.25) when plastic hinges form in the midspans ($M_{\text{midspan}} = M_p$). The level of the additional load for which this occurs is:
$$\frac{5}{32}P_y l + \frac{1}{4}P_a l = M_p \rightarrow \frac{5}{32}P_y l + \frac{1}{4}P_a l = \frac{3}{16}P_y l \rightarrow P_a = \frac{1}{8}P_y$$
Therefore, the collapse load is:
$$P_u = P_y + P_a = \frac{9}{8}P_y$$

- **Redistribution ratio at supports**: For the maximum load the structure can handle P_u, the moment of the supports accounting for yielding is:
$$M_C = M_p = \frac{5}{32}P_y l + \frac{1}{8}P_y \frac{1}{4} l = \frac{3}{16}P_y l$$
whereas ignoring yielding of the cross section the moment at C is:
$$M_C = \frac{3}{16}P_u l = \frac{9}{8}\frac{3}{16}P_y l$$
Therefore, the moment redistribution factor is:
$$\beta_b = \frac{\frac{3}{16}P_y l}{\frac{9}{8}\frac{3}{16}P_y l} = \frac{8}{9} = 0.889 \leq 1$$

In other words, the elastic analysis overestimates support moments, compared to a more realistic plastic analysis.

2.5.6 Code Provisions for Moment Redistribution (HKCC2013)

The Hong Kong Concrete Code (HKCC2013, Clause 9.9.1.1) [7, 2, 16] provides guidelines for moment redistribution in reinforced concrete design, introducing the **moment redistribution ratio** β_b, defined as the ratio of the actual (plastic) moment to the elastic moment. This parameter adjusts the **neutral axis depth limit** (x) and ensures sufficient ductility and rotational capacity at critical sections (e.g., supports or mid-spans) to accommodate redistribution while preventing excessive demand on concrete failure.

Neutral Axis Depth Limits

The maximum neutral axis depth (x_{max}) is adjusted based on the percentage of moment redistribution:

$$x_{\text{max}} = \begin{cases} 0.5d & \text{if } 0\,\% \leq \text{redistribution} \leq 10\,\% \\ (\beta_b - 0.4)d & \text{if } 10\,\% < \text{redistribution} \leq 30\,\% \text{ and } f_{cu} \leq 45\,\mathrm{N/mm^2} \\ (\beta_b - 0.5)d & \text{if } 10\,\% < \text{redistribution} \leq 30\,\% \text{ and } 45\,\mathrm{N/mm^2} < f_{cu} \leq 70\,\mathrm{N/mm^2} \end{cases}$$

These limits ensure the section remains under-reinforced, allowing plastic rotation. Specifically:

- $0\,\% \leq$ redistribution $\leq 10\,\%$: $x \leq 0.5d$, $\beta_b \geq 0.9$ (no significant redistribution).
- $10\,\% <$ redistribution $\leq 20\,\%$: $x \leq 0.4d$, $\beta_b = 0.8$.
- $20\,\% <$ redistribution $\leq 30\,\%$: $x \leq 0.3d$, $\beta_b = 0.7$.

Moment Resistance and Redistribution Limits

The moment of resistance at any section must be at least $70\,\%$ of the moment from elastic analysis, permitting a maximum reduction of $30\,\%$ in the peak elastic moment for concrete with $f_{cu} \leq 45\,\mathrm{N/mm^2}$. Consequently:

$$\beta_b \geq 0.7$$

While the code allows up to $30\,\%$ redistribution, a practical limit of $15\,\%$ is often recommended to balance ductility and safety, with $\leq 10\,\%$ typically considered in standard design practice.

Moment Resistance K' without Moment Redistribution

Refer again to the problem of a singly reinforced cross-section of Figure 2.7 and let us examine the limit case $K = K'$ when the moment resistance of the (singly reinforced) cross-section is exhausted. When $K > K'$, compression reinforcement is required, leading to a doubly reinforced section. $K = K'$ happens for $x = x_{\text{max}}$ or equivalently for $z = z_{min}$ since simple geometry dictates that:

$$z = d - \frac{s}{2} \quad \rightarrow \quad z_{\text{min}} = d - \frac{s_{\text{max}}(f_{cu})}{2}$$

Note that $x = x_{\text{max}}$ and $s = s_{\text{max}}$ are equivalent, since s is by definition (Equation 2.3) a percentage of x. Recall that this percentage depends on the concrete grade f_{cu}, and as a reminder, we denote s as a function of f_{cu}: $s = s(f_{cu})$.

We have already shown that from the moment equilibrium (Figure 2.7) we get:

$$M = F_{cc}z = 0.45 f_{cu} bs$$

In Section 2.3.3 we solved this equation for the lever arm, in order to design the cross-section. Instead, here we will examine what happens for at the limit $x = x_{\text{max}}$:

$$M = F_{cc}z = 0.45 f_{cu} b s_{\text{max}} \cdot z_{\text{min}}$$

For $f_{cu} \leq 45\,\mathrm{N/mm^2}$, $s_{\text{max}} = 0.9x_{\text{max}}$ (Equation 2.3) and $x_{\text{max}} = 0.5d$ (Table 2.3) hence:

$$z_{\text{min}} = d - \frac{s_{\text{max}}(f_{cu})}{2} = d - \frac{0.9x_{\text{max}}}{2} = d - \frac{0.45d}{2} = 0.775d$$

and

$$\begin{aligned} M_{cc,max} &= 0.45 f_{cu} b s_{max} \cdot z_{min} \\ &= 0.45 f_{cu} b(0.9x_{\text{max}}) \cdot z_{\text{min}} \\ &= 0.45 f_{cu} b 0.9(0.5d) \cdot 0.775d \\ &= 0.157 f_{cu} bd^2 \end{aligned}$$

This can be written as:

$$M_{cc} = K' bd^2 f_{cu} \tag{2.16}$$

which implies that $K' = 0.157$.
For $45\,\mathrm{N/mm^2} < f_{cu} \leq 70\,\mathrm{N/mm^2}$, $s = 0.8x$ (Equation 2.3) and $x_{\text{max}} = 0.4d$ (Table 2.3) hence:

$$z_{\text{min}} = d - \frac{s_{\text{max}}(f_{cu})}{2} = d - \frac{0.8x_{\text{max}}}{2} = d - \frac{0.32d}{2} = 0.68d$$

and

$$\begin{aligned}M_{cc,max} &= 0.45f_{cu}bs_{max}\cdot z_{min}\\ &= 0.45f_{cu}b(0.8x_{\text{max}})\cdot z_{\text{min}}\\ &= 0.45f_{cu}b0.8(0.4d)\cdot 0.68d\\ &= 0.121f_{cu}bd^2\\ &= K'f_{cu}bd^2\end{aligned}$$

which shows that $K' = 0.121$.

Table 2.4 presents the K' values for different concrete strength f_{cu} and negligible moment redistribution ($0\,\%$ to $10\,\%$, i.e., $\beta_b \leq 0.9$):

Concrete Strength	K'
$f_{cu} \leq 45\,\mathrm{N/mm^2}$	0.157
$45\,\mathrm{N/mm^2} < f_{cu} \leq 70\,\mathrm{N/mm^2}$	0.121
$70\,\mathrm{N/mm^2} < f_{cu} \leq 100\,\mathrm{N/mm^2}$	0.094

Table 2.4: K' values for moment redistribution of $0\,\%$ to $10\,\%$ ($\beta_b \leq 0.9$).

Maximum Moment Resistance Ratio K' with Moment Redistribution β_b

Moment redistribution reduces support moments and increases collapse load compared to elastic analysis. The code simplifies design by adjusting x limits based on β_b rather than requiring full elastic-plastic analysis. The moment redistribution factor β_b adjusts the maximum value of the neutral axis

$$x_{\text{max}} = (\beta_b - 0.4)d$$

therefore, it affects also the lever arm

$$z_{\text{min}} = d - 0.45(\beta_b - 0.4)d \tag{2.17}$$

and hence changes the value of K'.

For $f_{cu} \leq 45\,\mathrm{N/mm^2}$, $s = 0.9x$ the moment equilibrium gives:

$$\begin{aligned}M_{cc,max} &= F_{cc}\cdot z_{min}\\ &= 0.45f_{cu}bs_{max}\cdot z_{min}\\ &= 0.45f_{cu}b(0.9(\beta_b - 0.4)d)\cdot(d - 0.45(\beta_b - 0.4)d)\\ &= 0.405bd^2f_{cu}(\beta_b - 0.4)\cdot[1 - 0.45(\beta_b - 0.4)]\\ &= \left[0.405(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2\right]bd^2f_{cu}\end{aligned}$$

which shows that:

$$K' = 0.405(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2 \tag{2.18}$$

A lower β_b indicates greater redistribution. For example, a moment redistribution of $20\,\%$ corresponds to $\beta_b = 0.8$. Substituting $\beta_b = 0.8$ into Equation 2.18:

$$\begin{aligned}K' &= 0.405(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2\\ &= 0.405(0.8 - 0.4) - 0.18(0.8 - 0.4)^2 = 0.133\end{aligned}$$

Concrete Strength	K'
$f_{cu} \leq 45\,\mathrm{N/mm^2}$	$0.405(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2$
$45\,\mathrm{N/mm^2} < f_{cu} \leq 70\,\mathrm{N/mm^2}$	$0.357(\beta_b - 0.5) - 0.143(\beta_b - 0.5)^2$

Table 2.5: K' values for moment redistribution greater than $10\,\%$ ($\beta_b < 0.9$).

Thus, $K' = 0.133$, which is less than the value of $K' = 0.156$ typically used for no redistribution ($\beta_b \leq 0.9$). This reduction in K' reflects the effect of moment redistribution on the section's capacity.

Table 2.5 presents the K' calculation formulas for moment redistribution greater than $10\,\%$ ($\beta_b < 0.9$), adjusted for f_{cu}:

Table 2.6 summarizes the K' values for different moment redistribution levels and $f_{cu} \leq 45\,\mathrm{N/mm^2}$: Key parameters, including the moment capacity factor K' and lever arm ratio $z_{\min}/d$, vary with redistribution:

Redist. (%)	β_b	$\frac{x}{d}$	$\frac{z_{\min}}{d}$	K'	$\frac{d'}{d}$
≤ 10	≥ 0.9	0.5	0.775	0.157	0.185
$10 \to 20$	$0.8 \to 0.9$	0.4	0.820	0.133	0.150
$20 \to 30$	$0.7 \to 0.8$	0.3	0.865	0.104	0.112

Table 2.6: Design parameters for moment redistribution ($f_{cu} \leq 45\,\mathrm{N/mm^2}$).

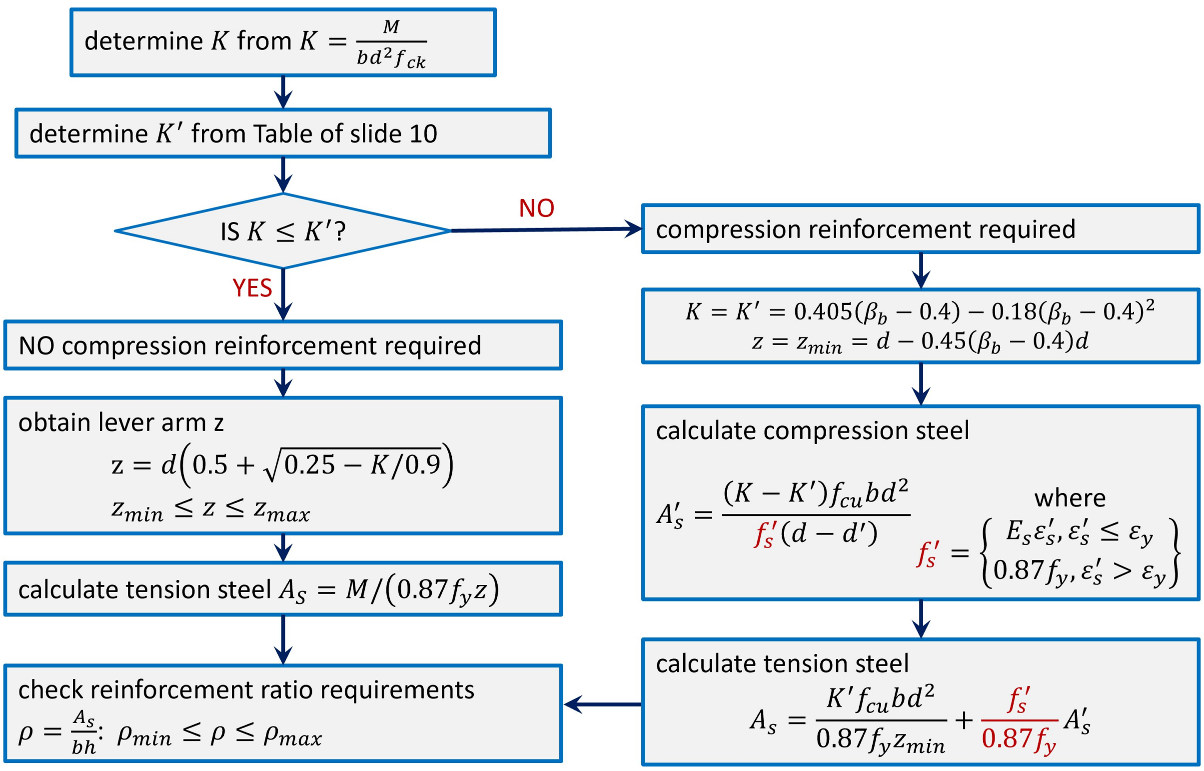

Figure 2.27: Flowchart of cross section design for flexure.

2.5.7 Summary of section design

- **Design parameter**

  1. Calculate $K = \dfrac{M}{f_{cu} b d^2}$
  2. Check K vs K'
     $$K = \begin{cases} K \le K', & \text{no compression reinforcement is required} \rightarrow \text{singly reinforced section} \\ K > K', & \text{compression steel is required, } z = z_{\text{min}} \rightarrow \text{doubly reinforced section} \end{cases}$$
     See Tables 2.3 and 2.6 for the appropriate value of K'.

- **Singly reinforced section**

  1. Calculate the lever arm $z = d\left(0.5 + \sqrt{0.25 - \dfrac{K}{0.9}}\right)$
     if $z > z_{max}$ take $z = z_{max}$; this occurs for low levels of moment.
  2. Calculate required tension steel $A_s = \dfrac{M}{0.87 f_y z}$
  3. Choose bars and check the reinforcement ratio $\rho_{min} \le \rho = \dfrac{A_s}{b \cdot h} \le \rho_{max}$

- **Doubly reinforced section**

  1. For $K = K'$ it follows that $x = x_{max}$ and $z = z_{min}$ for example $x_{max} = 0.5d$ and $z_{min} = 0.775d$.
  2. Calculate the depth of stress block $s = 2(d - z_{min})$; then $x = \dfrac{s}{0.9}$
  3. Check yielding of compression steel $\frac{d'}{x} \begin{cases} d'/x \le 0.38 \rightarrow \varepsilon'_s \ge \varepsilon_y \rightarrow f'_s = 0.87 f_y \\ d'/x > 0.38 \rightarrow \varepsilon'_s < \varepsilon_y \rightarrow f'_s = E_s \varepsilon'_s \end{cases}$
  4. Calculate $A'_s = \dfrac{(K - K') f_{cu} b d^2}{f'_s (d - d')}, \quad A_s = \dfrac{K' b d f_{cu}}{0.87 f_y z_{\text{min}}} + \left(\dfrac{f'_s}{0.87 f_y}\right) A'_s$
  5. Choose bars and check the reinforcement ratio $\rho_{min} \le \rho = \dfrac{A_s}{b \cdot h} \le \rho_{max}$

Notes:

- Recall the effect of f_{cu} (see Table 2.3) and of moments redistribution β_b on the formulae (Table 2.6 or Equation 2.18) and in particular on K' and x_{max} limits

- Handling cross-sections with both positive and negative moments:
  Design separately for each moment combination. For example assume that

  – for one load combination with negative moment: 5T25 at the top and 3T25 at the bottom;
  – for another with positive moment: 4T25 at the bottom.

  Supply the maximum reinforcement at each location (e.g., 5T25 at the top, 4T25 at the bottom) to cover all cases.

2.6 Design of Flanged Sections

2.6.1 Overview

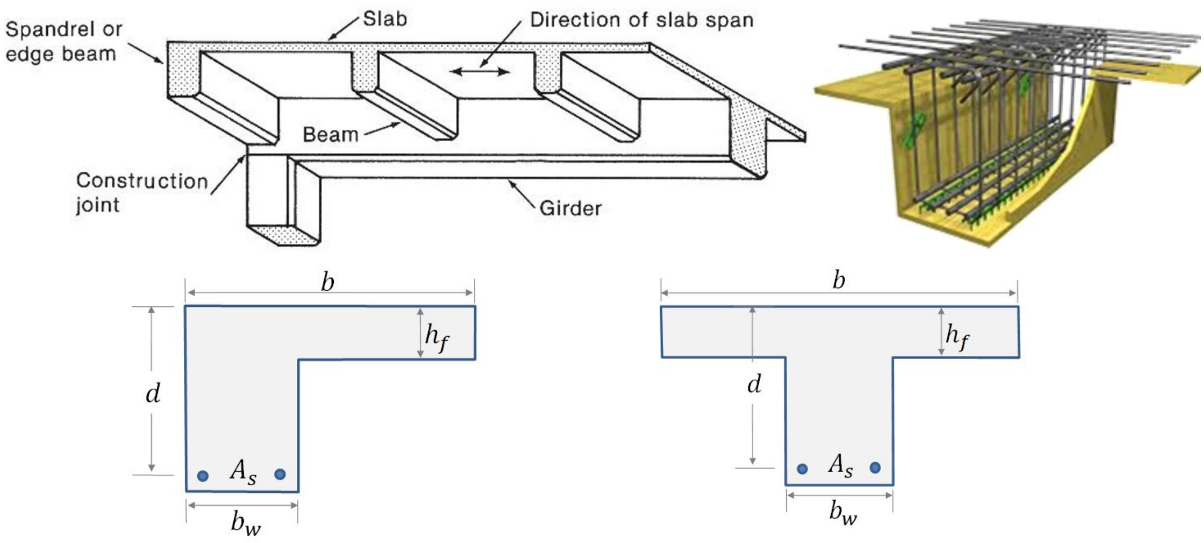

Figure 2.28: RC beams connected monolithically to slabs (top) behave as flanged-sections (bottom) under positive (sagging) moments.

Beams in real structures are typically integrated with slabs, forming flanged sections such as T-beams or L-beams, rather than functioning as isolated rectangular sections.

- A T-section forms when a beam is cast monolithically with a slab (Figure 2.28), with the slab acting as the flange and the beam as the web.
- An L-section forms at the edge of a slab (Figure 2.28), where the beam extends above or below the slab.

These flanged beams, where the flange is part of the floor slab, raise the question of determining the effective width of the slab to be considered in design. The effective width, denoted as b, plays a critical role in simplifying the analysis of such beams by allowing to treat the flanged sections independently from the slab for design purposes.

- **Flange:** The horizontal part of the beam, typically the slab portion, with dimensions $b \times h_f$.
- **Web:** The vertical part of the cross-section has dimensions $b_w \times d$. The portion of the web below the flange has dimensions $b_w \times (d - h_f)$.
- **Wings:** The portions of the flange extending outside the web, with dimensions $(b - b_w) \times h_f$.
- **Effective Width (b):**
  - In this course, b will be given.
  - In practice, b is usually determined as a portion of the effective span, depending on the moment diagram and, consequently, on the structural system (e.g., supported, continuous, or cantilever beam).

In a continuous beam (e.g., Figure 2.29), the bending moment varies along the length. Typically,

- at mid-spans (section A-A), sagging moments cause tension at the bottom and compression at the top, making the flange (slab) part of the compression zone.
- at supports (section B-B), hogging moments cause tension at the top and compression at the bottom, making the section behave as a rectangular section (since the flange is in tension and hence ignored).

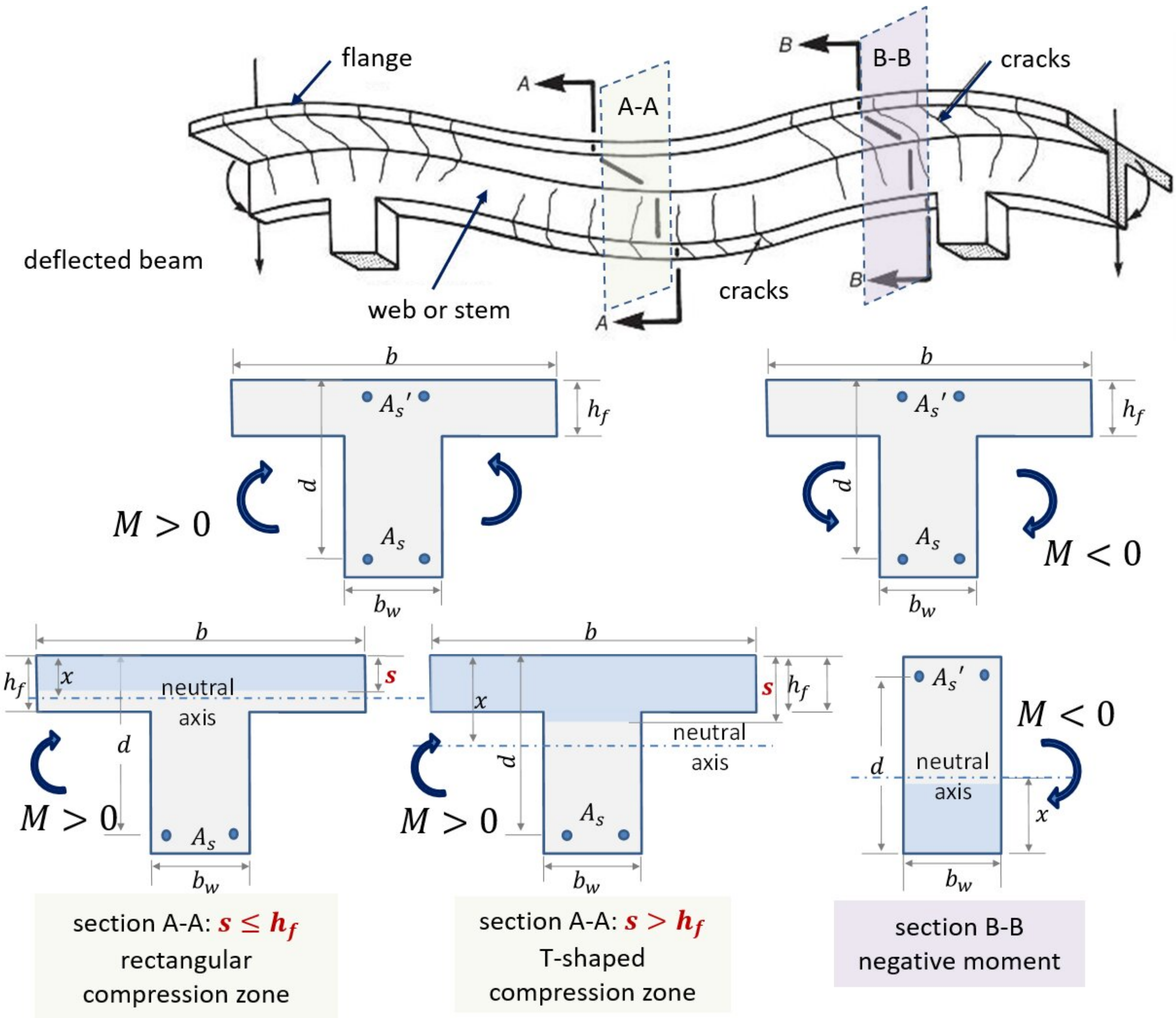

Figure 2.29: At mid-spans (positive bending moments) the flanged is under compression, at supports (negative bending moments) the cross section behaves as a rectangular section with the width of the web

2.6.2 Design Procedure

The key assumptions for designing flanged sections are the same used previously (Section 2.1.1), including the neglect of concrete tensile strength and the adoption of the same equivalent stress block (Section 2.2.3). We focus on singly reinforced sections initially, then address doubly reinforced cases. The first step in designing a flanged section under sagging moments (tension at the bottom) is the position of the stress block relative to the flange depth h_f.

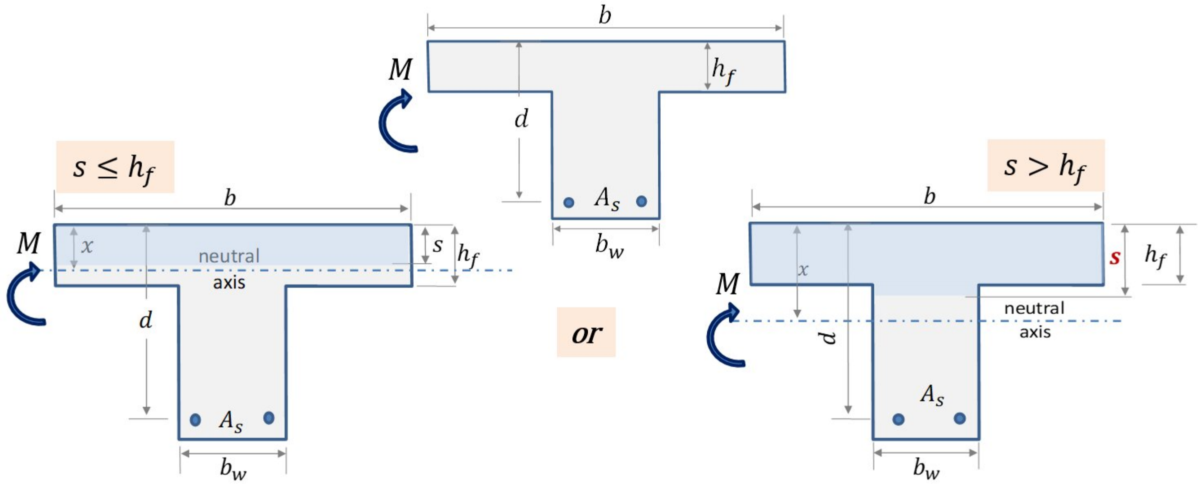

Figure 2.30: Flanged section in bending. Left: $s \leq h_f$, right: $s > h_f$

Key Question: Position of the Stress Block with respect to the flange
In a flanged section subjected to a positive (sagging) moment tension develops at the bottom of the section and compression at the top. The depth of the stress block s (see Table 2.3) may either lie within the flange ($s \leq h_f$) or extend below it ($s > h_f$). To determine the position of the stress block relative to the flange depth h_f, we analyze the limiting case where $s = h_f$, corresponding to the scenario in which the entire flange is under uniform compression. Specifically,

- The compressive force of the flange is $F_f = 0.45 f_{cu} b h_f$.
- The moment resistance of the flange is $M_f = 0.45 f_{cu} b h_f (d - h_f/2)$.

Compare M_f with the design moment M:

- If $M_f \geq M$, the stress block lies within the flange: $s \leq h_f$.
- If $M_f < M$, the stress block extends below the flange: $s > h_f$.

The design problem of each case differs.

2.6.3 Case 1: Compression zone within flange $s \leq h_f$ (Figure 2.31)

When the stress block lies within the flange of a T-beam or L-beam, the region below the neutral axis undergoes tension, rendering it ineffective, as concrete's tensile resistance is neglected in design. Consequently, the beam can be analyzed and designed as an equivalent rectangular section using the flange's width. This approach simplifies the analysis by assuming the entire compression zone is confined within the flange. Thus, the design of the flanged section effectively reduces to that of an equivalent rectangular section with dimensions $b \times d$. Since we have already addressed this design case in Section 2.3.3, the description of the solution procedure is kept concise here:

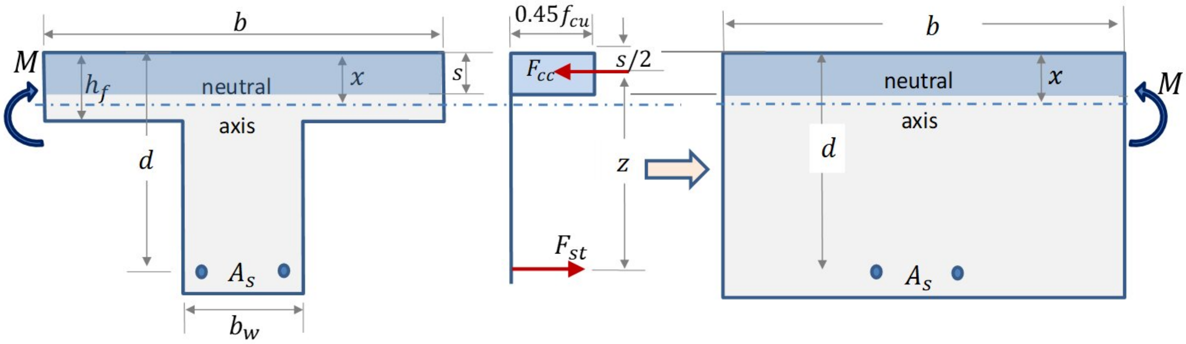

Figure 2.31: Design of flanged section with $s \leq h_f$

Design summary/premise when $s \leq h_f$:

Treat the flanged section as a rectangular section with dimensions $b \times d$ (Section 2.3.3):

- Calculate bending moment ratio: $K = \dfrac{M}{bd^2 f_{cu}}$.
- Determine the lever arm: $\frac{z}{d} = 0.5 + \sqrt{0.25 - \frac{K}{0.9}}$,
- Calculate required tension steel: $A_s = \dfrac{M}{0.87 f_y z}$
- Choose bars and check the reinforcement ratio: $\rho_{\text{min}} \leq A_s/(b_w h) \leq \rho_{\text{max}}$.

2.6.4 Case 2: Compression zone below flange $s > h_f$ (Figure 2.32)

If the stress block extends below the flange, the entire flange is in compression, but additional compression is needed from the web. We split the design moment M into two parts, m_1: the wings (of the flange) and m_2 the web contribution (see Figure 2.32):

- $M = m_1 + m_2$: Total moment applied on the section.
- m_1: Moment resistance of wings (flange minus web) generating a steel demand of A_{s1}.
- $m_2 = M - m_1$: Remaining moment resisted by the web, corresponding steel A_{s2}.
- $A_s = A_{s1} + A_{s2}$: Total steel demand is the sum.

Moment Resisted by the Wings (m_1)

The wings are the portions of the flange outside the web (width $b - b_w$, depth h_f). Since the entire flange is in compression we can calculate the compressive force of the wings and the corresponding moment:

- Compressive force in the wings (F_{cc1}):

$$F_{cc1} = 0.45 f_{cu}(b - b_w)h_f$$

- Lever arm is known as the point of application of the F_{cc1} force is known:

$$z_1 = d - \frac{h_f}{2}$$

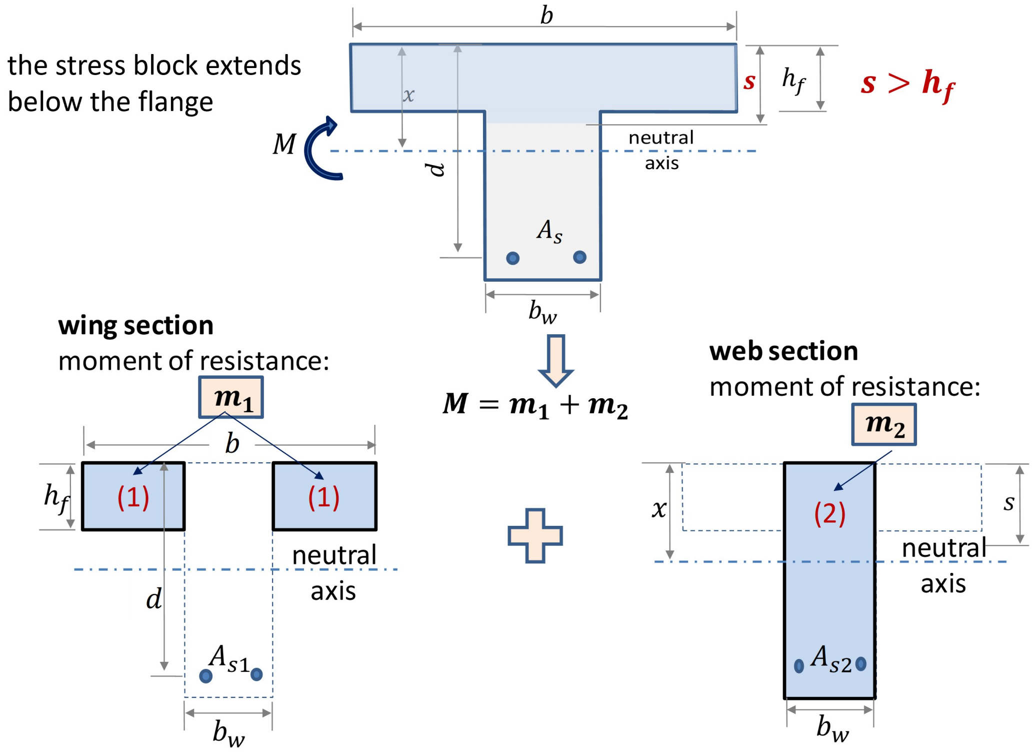

Figure 2.32: Design a flanged section with $s > h_f$

- Moment m_1:

$$m_1 = F_{cc1}\left(d - \frac{h_f}{2}\right) = 0.45 f_{cu}(b - b_w)h_f\left(d - \frac{h_f}{2}\right)$$

The corresponding tension steel area A_{s1} to balance m_1:

$$A_{s1} = \frac{m_1}{0.87 f_y\left(d - \frac{h_f}{2}\right)}$$

Moment Resisted by the Web ($m_2 = M - m_1$)- First-Principles Method

Premise: Treat the web as a rectangular section ($b_w \times d$), subjected to the remaining moment $m_2(= M - m_1)$ and follow the design procedure of an equivalent rectangular section to solve for z and A_{s2}. Since this is a case we have covered in Section 2.3.3, we keep the description of the solution procedure very short here:

1. Calculate $K = m_2/(b_w d^2 f_{cu}) \rightarrow$ if $K \leq K'$, the section remains singly reinforced.

2. Determine the lever arm: $z_2 = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right)$

3. Calculate the additional steel area:

$$A_{s2} = \frac{m_2(= M - m_1)}{0.87 f_y z_2}$$

4. Total required steel is: $A_s = A_{s1} + A_{s2}$.

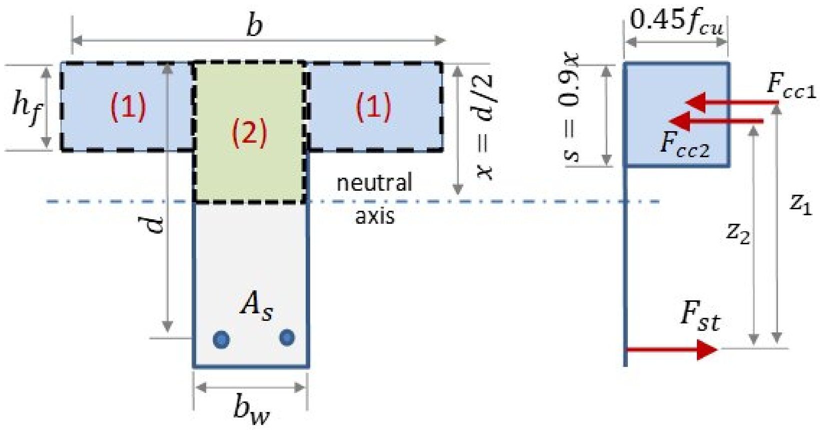

Figure 2.33: Design a flanged section with $s > h_f$

5. Choose bars and check the reinforcement ratio considering only the area of the web:

$$\rho_{\text{min}} \leq \frac{A_s}{b_w \cdot h} \leq \rho_{\text{max}}$$

.

Moment Resisted by the Web ($m_2 = M - m_1$)- Code Method

Premise: The Code method, as outlined in [Clause 6.1.2.4 (d)], provides a conservative design approach for flanged sections, such as T-beams and L-beams, by assuming the neutral axis depth at its maximum, $x = x_{\text{max}}$. Recall that the specific value of x_{max} depends on f_{cu} see Table 2.3. For instance, for a characteristic concrete strength $f_{cu} \leq 45\,\text{N/mm}^2$,$x_{\text{max}} = 0.5d$,

$$z_2 = \quad d - 0.45x_{\text{max}} = d - 0.45 \cdot 0.5d = 0.775d$$

Regardless of the specific value of x_{max}, the central assumption of the Code method is that $x = x_{\text{max}}$ which implies also that $z = z_{\text{min}}$ and ensures that the design is conservative.

The next step is to calculate the maximum moment resistance of the flanged section without compression steel $M_{c,\text{max}}$. Breaking again the flanged cross section into two parts, the wings and the web, it holds that the force of:

- **wings area 1** : $(b - b_w)h_f \rightarrow F_{cc1} = 0.45f_{cu}(b - b_w)h_f$

- **web area 2** : $b_w s \rightarrow \qquad F_{cc2} = 0.45f_{cu}b_w(0.45d) = 0.2f_{cu}b_w d$

The maximum moment resistance of the section without compression steel is:

$$M_{c,\text{max}} = F_{cc1}z_1 + F_{cc2}z_2 = 0.45f_{cu}(b - b_w)h_f\left(d - \frac{h_f}{2}\right) + 0.156f_{cu}b_w d^2$$

which can also be written concisely as:

$$M_{c,\text{max}} = \beta_f f_{cu} b d^2 \tag{2.19}$$

in which:

$$\beta_f = 0.45\frac{h_f}{d}\left(1 - \frac{b_w}{b}\right)\left(1 - \frac{h_f}{2d}\right) + K'\frac{b_w}{b} \tag{2.20}$$

and K' depending on f_{cu} ($K' = K'(f_{cu})$) according to Table 2.3.

Analytical derivations
Here is the derivation from the force and lever arm contributions to the standard non-dimensional moment capacity expression used in BS 8110 and HKCC2013 [7]. The maximum moment capacity provided by the concrete compression block is given by

$$M_{c,\max} = F_{cc1}z_1 + F_{cc2}z_2$$

where the compression forces and lever arms are:

- Flange compression force: $F_{cc1} = 0.45f_{cu}(b - b_w)h_f$
- Web compression force: $F_{cc2} = 0.45f_{cu}b_w(0.9s) = 0.45f_{cu}b_w(0.45d) = 0.2f_{cu}b_w d$ where $0.45 \cdot 0.45 = 0.2025$ with 0.2025 rounded to 0.2 for simplicity.
- Flange lever arm: $z_1 = d - \frac{h_f}{2}$
- Web lever arm: $z_2 = 0.775d$

Substituting these expressions into $M_{c,\max}$:

$$\begin{aligned} M_{c,\max} &= 0.45f_{cu}(b - b_w)h_f\left(d - \frac{h_f}{2}\right) + 0.2f_{cu}b_w d \cdot 0.775d \\ &= 0.45f_{cu}(b - b_w)h_f\left(d - \frac{h_f}{2}\right) + 0.156f_{cu}b_w d^2 \end{aligned}$$

To express $M_{c,\max}$ in non-dimensional form divide both sides by $f_{cu}bd^2$:

$$\begin{aligned} \frac{M_{c,\max}}{f_{cu}bd^2} &= \frac{0.45f_{cu}(b - b_w)h_f\left(d - \frac{h_f}{2}\right)}{f_{cu}bd^2} + \frac{0.156f_{cu}b_w d^2}{f_{cu}bd^2} \\ &= 0.45 \cdot \frac{b - b_w}{b} \cdot \frac{h_f}{d} \cdot \left(1 - \frac{h_f}{2d}\right) + 0.156 \cdot \frac{b_w}{b} \end{aligned}$$

Therefore, with the aid of the coefficient β_f:

$$\beta_f = 0.45\frac{h_f}{d}\left(1 - \frac{b_w}{b}\right)\left(1 - \frac{h_f}{2d}\right) + K'\frac{b_w}{b}$$

the maximum moment capacity can be expressed as:

$$M_{c,\max} = \beta_f f_{cu}bd^2$$

where K' is the limiting web contribution factor depending on concrete strength:

- $K' = 0.156$ if $f_{cu} \geq 45\,\mathrm{N/mm^2}$
- $K' = 0.120$ if $45 < f_{cu} \leq 70\,\mathrm{N/mm^2}$
- $K' = 0.094$ if $70 < f_{cu} \leq 100\,\mathrm{N/mm^2}$

Compare $M_{c,\max}$ (Equation 2.19) with the design moment M

- if $M > M_{c,\max}$ compression steel is needed, refer to Section 2.6.5.
- if $M \leq M_{c,\max}$ compression steel is not required, the section is singly reinforced.

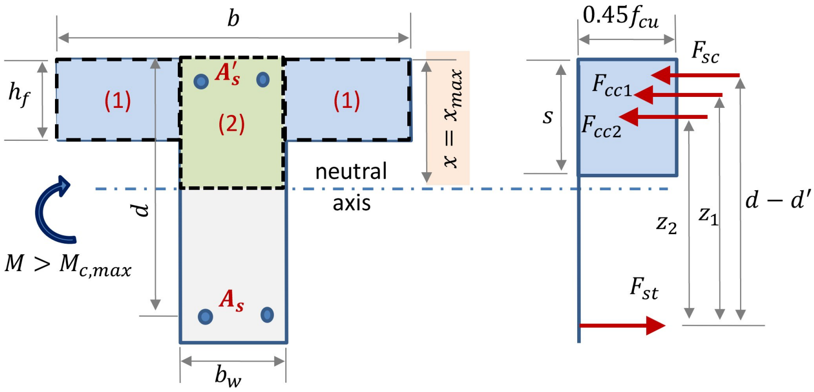

Figure 2.34: Design a flanged section with $s > h_f$

- The total required area of tension reinforcement (A_s) can be determined by taking moments about the centroid of the compression force in the flange (F_{cc1}). The maximum moment capacity ($M_{u\max}$) is given by:

$$M_{u\max} = F_{st}z_1 - F_{cc2}\left(\frac{s}{2} - \frac{h_f}{2}\right) = 0.87f_yA_s\left(d - \frac{h_f}{2}\right) - 0.2f_{cu}b_wd\left(0.45d - \frac{h_f}{2}\right)$$

- From this, the required area of tension reinforcement (A_s) can be derived as:

$$A_s = \frac{M + k_1f_{cu}b_wd(k_2d - h_f)}{0.87f_y(d - 0.5h_f)}$$

- where k_1 and k_2 are coefficients dependent on the characteristic concrete strength (f_{cu}), as provided in Clause 6.1.2.4(d) of the code. The values are summarized in the following table:

$f_{cu} \leq \mathrm{N/mm^2}$	k_1	k_2
$45\,\mathrm{N/mm^2}$	0.100	0.45
$70\,\mathrm{N/mm^2}$	0.072	0.32
$100\,\mathrm{N/mm^2}$	0.054	0.24

Table 2.7: Coefficients k_1 and k_2 for flanged sections (Clause 6.1.2.4(d)).

2.6.5 Design of Doubly Reinforced Flanged Sections

The need for compression steel in a flanged section arises only after the resistance of both the wings and the web has been fully utilized, resulting in $s > h_f$ and $x = x_{\max}$. This implies that the design of a doubly reinforced flanged section follows the design of a singly reinforced flanged section, up to the point where the contribution of the web is calculated. The subsequent steps then depend on the chosen method—either the First-Principles Method or the Code Method—as outlined below:

Doubly Reinforced T-Section - First Principles Method

Premise: Treat the web as a rectangular section with dimensions $b_w \times d$, subjected to moment $m_2 = M - m_1$ and follow the design procedure of an equivalent rectangular section which needs compression steel to solve for A_{s2} and A'_s.

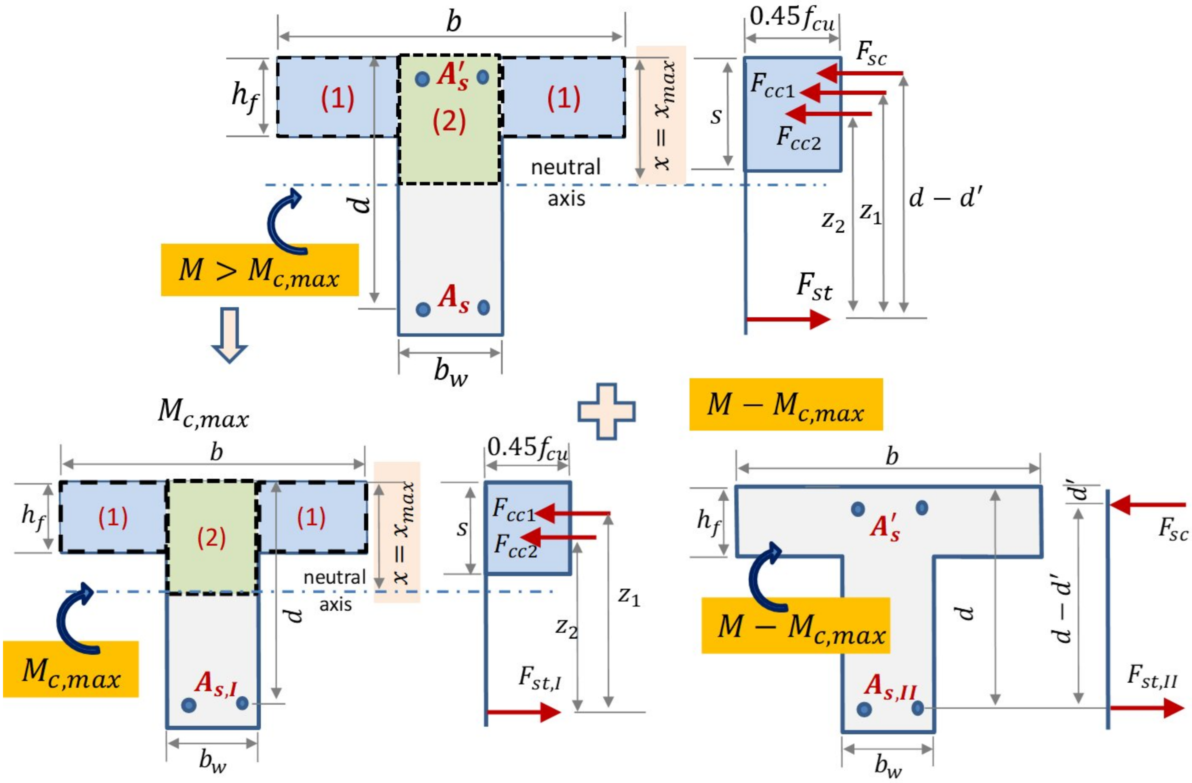

Figure 2.35: Design a doubly reinforced flanged section

In particular, the calculation of the resistance of the wings and the corresponding required steel A_{s1} is identical. For the remaining moment $M - m_1$ calculate the resistance of the web, treated as a rectangular section with width b_w and effective depth d. Following the design procedure of a doubly reinforced rectangular section (Section 2.4.1), first calculate $K = \frac{m_2}{b_w d^2 f_{cu}}$. Then compare K with K':

- if $K > K'$, the section needs compression steel and it is a doubly reinforced. $x = x_{max}$, or equivalently that $z = z_{min}$ reinforced capacity at $x = x_{max}$).
- **The remaining steps are as described in Section 2.4.1** for rectangular doubly reinforced sections.
- Choose bars and check the reinforcement ratio considering only the tension steel and the area of the web:

$$\rho_{\min} \leq \frac{A_s(= A_{s1} + A_{s2})}{b_w \cdot h} \leq \rho_{\max}$$

.

Doubly Reinforced T-Section - Code Method

Compare $M_{c,\mathbf{max}}$ (from Equation 2.19) with the design moment M

- if $M \leq M_{c,\text{max}}$, the section remains singly reinforced and the design procedure
- if $M > M_{c,\text{max}}$ the section becomes doubly reinforced since compression steel A_s' is needed to resist the surplus design moment ($M - M_{c,\text{max}}$)
- Taking moments about the centroid of the tension steel (see Figure 2.35 bottom right):

$$M - M_{c,\text{max}} = f_y' A_s' (d - d')$$

  where the stress of the compression steel f_y' (Equation 2.1) is $f_y' = 0.87 f_y$ if it has has yielded ($\varepsilon_s' = \varepsilon_{cu} \frac{x-d'}{x} > \varepsilon_y$) or $f_s' = E_s \varepsilon_s'$ otherwise.

- Therefore the compression still required (assuming yielding) is:

$$A_s' = \frac{M - M_{c,\text{max}}}{0.87 f_y (d - d')} = \frac{M - \beta_f f_{cu} b d^2}{0.87 f_y (d - d')} \tag{2.21}$$

- From the revised force equilibrium condition (which now includes the force of the compression steel) we calculate the total tension reinforcement required in the doubly reinforced section:

$$F_{st} = F_{cc1} + F_{cc2} + F_{sc} \rightarrow 0.87 f_y A_s = 0.45 f_{cu} b_w 0.45 d + 0.45 f_{cu} (b - b_w) h_f + 0.87 f_y A_s' \rightarrow$$

$$A_s = \frac{0.2 f_{cu} b_w d + 0.45 f_{cu} (b - b_w) h_f}{0.87 f_y} + A_s' \tag{2.22}$$

- Choose bars and check the reinforcement ratio considering only the tension steel and the area of the web:

$$\rho_{\text{min}} \leq \frac{A_s}{b_w \cdot h} \leq \rho_{\text{max}}$$

2.6.6 Example 1: Checking a flanged section with $s \leq h_f$

Determine the ultimate moment of resistance of the T-beam section **given**:

- Geometry $b = 800\,\text{mm}$, $b_w = 200\,\text{mm}$, $d = 420\,\text{mm}$, $h_f = 150\,\text{mm}$.
- Characteristic material strengths $f_{cu} = 30\,\text{MPa}$, $f_y = 500\,\text{MPa}$.
- Reinforcement: tension steel area $A_s = 1470\,\text{mm}^2$.

Solution steps:

1. **Assumptions:**
   - the stress block depth lies within the flange $s \leq h_f$
   - the reinforcement is strained to the yield: $f_s = 0.87 f_y$.
2. Based on theses assumptions, we can directly determine the compression zone depth from the force equilibrium:

$$F_{cc} = F_{st} \rightarrow 0.45 f_{cu} b s = 0.87 f_y A_s \rightarrow$$

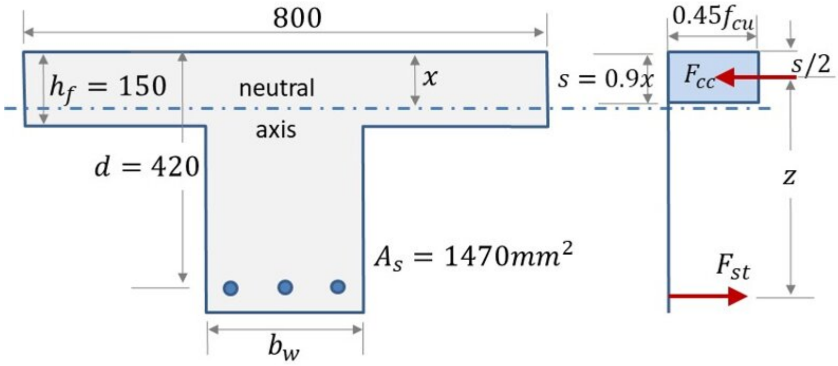

Figure 2.36: Example 1:Checking of flanged section.

$$s = \frac{0.87 f_y A_s}{0.45 f_{cu} b} = \frac{0.87 \cdot 500 \cdot 1470}{0.45 \cdot 30 \cdot 800} = 59\,\mathrm{mm} < h_f = 150\,\mathrm{mm}.$$

Since $s < h_f$ the compression zone is indeed within flange. Also, for this compression zone depth, the tension steel has yielded as assumed since:

$$x = \frac{s}{0.9} = \frac{59}{0.9} = 66\,\mathrm{mm} < x_{bal} = 0.617d = 315\,\mathrm{mm}$$

3. Lever arm:

$$z = d - \frac{s}{2} = 420 - \frac{59}{2} = 391\,\mathrm{mm}$$

4. The moment capacity of the section is determined taking moments about A_s:

$$M_R = F_{cc} \cdot 0.45 f_{cu} f_{bs} z = 0.45 \cdot 30 \cdot 800 \cdot 59 \cdot 391 \cdot 10^{-6} = 249.1\,\mathrm{kN\,m}$$

2.6.7 Example 2: Design a flanged section with $s > h_f$

Design a T-beam section **given**:

- Geometry $b = 400\,\mathrm{mm}$, $b_w = 200\,\mathrm{mm}$, $d = 350\,\mathrm{mm}$, $h_f = 100\,\mathrm{mm}$.
- Characteristic material strengths $f_{cu} = 30\,\mathrm{MPa}$, $f_y = 500\,\mathrm{MPa}$.
- Design moment: $M = 170\,\mathrm{kN\,m}$.

Solution steps:

1. Moment of resistance of the flange is:

$$M_f = 0.45 f_{cu} b h_f \left(d - \frac{h_f}{2}\right) = 0.45 \cdot 30 \cdot 400 \cdot 100 \left(350 - \frac{100}{2}\right) \cdot 10^{-6} \rightarrow$$

$$M_f = 162\,\mathrm{kN\,m} < 170\,\mathrm{kN\,m}$$

. Since $M_f < M$ the compression zone extends below the flange ($s > h_f$).

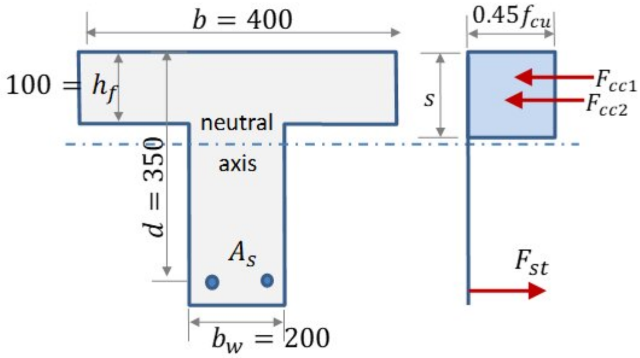

Figure 2.37: Design a flanged section with $s > h_f$

2. Moment resistance of wings and corresponding steel area:

$$m_1 = 0.45 f_{cu} (b - b_w) h_f \left(d - \frac{h_f}{2} \right) = 0.45 \cdot 30 \cdot (400 - 200) \cdot 100\,(350 - 50) \cdot 10^{-6}$$

$$m_1 = 81\,\mathrm{kN\,m}$$

$$A_{s1} = \frac{m_1}{0.87 f_y \left(d - \frac{h_f}{2} \right)} = \frac{81 \cdot 10^6}{0.87 \cdot 500\,(350 - 0.5 \cdot 100)} = 621\,\mathrm{mm}^2$$

3. Web **(First Principles solution)**:
   Moment applied on the web: $m_2 = M - m_1 = 170 - 81 = 89\,\mathrm{kN\,m}$.

   - $K = \frac{m_2}{b_w d^2 f_{cu}} = \frac{(170-81)\cdot 10^6}{200 \cdot 350^2 \cdot 30} = 0.121 < 0.156 = K'$
   - $z_2 = d \left(0.5 + \sqrt{0.25 - \frac{K=0.121}{0.9}} \right) = 294\,\mathrm{mm} > z_{min} = 0.775d = 271\,\mathrm{mm}$
   - $A_{s2} = \frac{m_2}{0.87 f_y z} = \frac{89 \cdot 10^6}{0.87 \cdot 500 \cdot 294} = 696\,\mathrm{mm}^2$.

4. Total area of tension steel is:

$$A_s = A_{s1} + A_{s2} = 621 + 696 = 1317\,\mathrm{mm}^2 \rightarrow \text{ Provide:3T25 } (A_s = 1472\,\mathrm{mm}^2)$$

5. Check the **steel ratio** (Figure 2.16):

$$\rho_{min} = 0.3\,\% < \rho = \frac{A_s}{b_w h} = \frac{1472}{200 \cdot 400} = 1.8\,\% < \rho_{max} = 2.5\,\% \rightarrow \text{OK.}$$

2.6.8 Example 3: Design of a Flanged Section in Bending (T-Beam) for all positive moments

Geometry and Material Properties

Consider a flanged T-beam section with the following dimensions:
$b = 400\,\mathrm{mm}$, $b_w = 200\,\mathrm{mm}$, $h_f = 100\,\mathrm{mm}$, $d = 350\,\mathrm{mm}$, $h = 400\,\mathrm{mm}$. and material strengths: $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$.
We design the section for **all positive (sagging) bending moment values**.

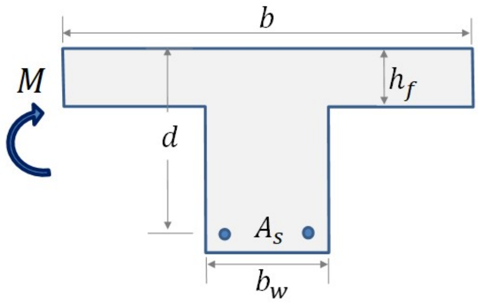

Figure 2.38: Design a flanged section for all positive moments.

Case 1: Very Small Bending Moments, $0 \le M \le M_{\min}$

Assume a very small moment, $M \to 0$. **Solution steps**

1. Calculate the bending moment ratio K

$$K = \frac{M}{b_w d^2 f_{cu}} = \frac{0}{200 \cdot 350^2 \cdot 30} = 0 < K'.$$

2. Calculate the lever arm z.

$$z = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) = d\left(0.5 + \sqrt{0.25 - 0}\right) = d > z_{\max} = 0.95d = 333\,\mathrm{mm}.$$

3. Calculate the required tension steel area A_s.

$$A_s = \frac{M}{0.87 f_y z} = \frac{0 \cdot 10^6}{0.87 \cdot 500 \cdot 333} = 0\ \mathrm{mm}^2.$$

4. Check if the steel ratio ρ satisfies the minimum reinforcement requirement. If not, provide the minimum reinforcement.

$$\rho = \frac{A_s}{b_w h} = 0 < \rho_{\min} = 0.3\%.$$

So we need to provide the minimum reinforcement.

Minimum Reinforcement
For a flanged cross section:

$$\rho_{\min} = 0.3\% \quad \Rightarrow \quad A_{s,\min} = \rho_{\min} b_w h = 0.003 \cdot 200 \cdot 400 = 240\,\mathrm{mm}^2.$$

For the pseudo-rectangular section using the full flange width b:

$$A_{s,\min} = \rho_{\min} bh = 0.003 \cdot 400 \cdot 400 = 480\,\mathrm{mm}^2.$$

Provide tension steel: $2T18 \quad (A_s \approx 509\,\mathrm{mm}^2)$.

M_{min}: Moment of Resistance of Minimum Reinforcement

M_{min} is the moment of resistance corresponding to the minimum reinforcement (here, $2T18$). Its calculation is an inverse (capacity estimation) problem.
Calculation assumptions: .

- The stress block lies entirely within the flange: $s \leq h_f$.
- The tension reinforcement yields: $f_s = 0.87 f_y$.

Solution steps

1. Force Equilibrium: Let s be the depth of the equivalent rectangular stress block. Force equilibrium gives
$$F_{cc} = F_{st} \quad \Rightarrow \quad 0.45 f_{cu}\, b\, s = 0.87 f_y A_s.$$
Solving for s:
$$s = \frac{0.87 f_y A_s}{0.45 f_{cu} b} = \frac{0.87 \cdot 500 \cdot 509}{0.45 \cdot 30 \cdot 400} = 41\,\text{mm} < h_f = 100\,\text{mm}.$$
Thus, the stress block lies within the flange.

2. *Lever Arm and Yield Check* The lever arm is
$$z = d - \frac{s}{2} = 350 - \frac{41}{2} \approx 330\,\text{mm}.$$

3. Moment of Resistance: Taking moments about the centroid of A_s, the moment of resistance is
$$M_{\text{min}} = F_{cc}\, z = 0.45 f_{cu} b s\, z = 0.45 \cdot 30 \cdot 400 \cdot 41 \cdot 330 \cdot 10^{-6} \approx 24.3\,\text{kN m}.$$

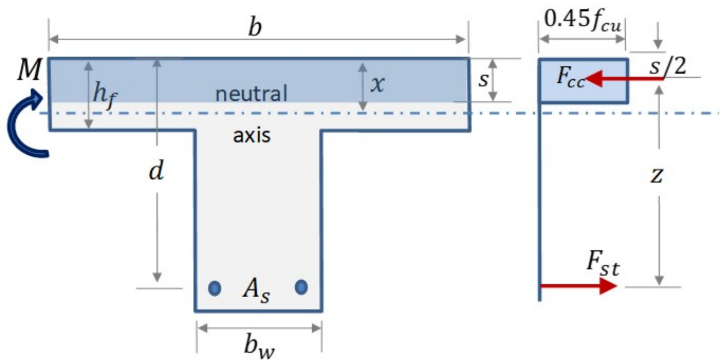

Figure 2.39: Design a flanged section for all positive moments.

Case 2: $M_{\text{min}} \leq M \leq M_f$ (Stress Block in Flange, $s \leq h_f$)

Moment M_f (when $s = h_f$) and the corresponding required steel area A_s are calculated as follows:
Solution steps

1. The compressive force when the stress block just reaches the bottom of the flange ($s = h_f$) is
$$F_{cc} = 0.45 f_{cu} b h_f = 0.45 \cdot 30 \cdot 400 \cdot 100 = 540\,\text{kN}.$$

2. The corresponding moment of resistance M_f is calculated by taking moments about the tension steel level:

$$M_f = 0.45 f_{cu} b h_f \left(d - \frac{h_f}{2}\right) = 0.45 \cdot 30 \cdot 400 \cdot 100 \left(350 - \frac{100}{2}\right) \cdot 10^{-6} = 162\,\mathrm{kN\,m}.$$

   For $M < M_f$, the stress block remains within the flange, i.e. $s < h_f$.

3. The bending moment ratio is

$$K = \frac{M_f}{b_w d^2 f_{cu}} = \frac{162 \cdot 10^6}{200 \cdot 350^2 \cdot 30} = 0.110 < K' = 0.156.$$

4. Hence the lever arm is

$$z = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) \approx 300\,\mathrm{mm}.$$

   Check:

$$z_{\min} = 0.775d < z < z_{\max} = 0.95d.$$

5. Required steel area:

$$A_s = \frac{M_f}{0.87 f_y z} = \frac{162 \cdot 10^6}{0.87 \cdot 500 \cdot 300} \approx 1241\,\mathrm{mm}^2.$$

   Provide:

$$4T20 \quad (A_s \approx 1257\,\mathrm{mm}^2).$$

   Steel ratio:

$$\rho = \frac{A_s}{b_w h} = \frac{1257}{200 \cdot 400} \approx 1.57\%,$$

   check:

$$\rho_{\min} = 0.3\% < \rho < \rho_{\max} = 2.5\%.$$

Case 3: $M_f \leq M \leq M_{c,\max}$ **(Stress Block Extends Below Flange)**

For $M > M_f = 162\,\mathrm{kN\,m}$, the stress block extends below the flange ($s > h_f$). The compression in the flange (wings) and the web are treated separately.

- Moment Resistance of Wing Sections:

$$\begin{aligned} m_1 &= F_{cc1}\left(d - \frac{h_f}{2}\right) \\ &= 0.45 f_{cu}(b - b_w) h_f \left(d - \frac{h_f}{2}\right) \\ &= 0.45 \cdot 30 \cdot (400 - 200) \cdot 100 \left(350 - \frac{100}{2}\right) \cdot 10^{-6} \\ &= 81\,\mathrm{kN\,m}. \end{aligned}$$

- The corresponding reinforcement area for the wings is

$$A_{s1} = \frac{m_1}{0.87 f_y \left(d - \frac{h_f}{2}\right)} = \frac{81 \cdot 10^6}{0.87 \cdot 500\,(350 - \frac{100}{2})} \approx 621\,\mathrm{mm}^2.$$

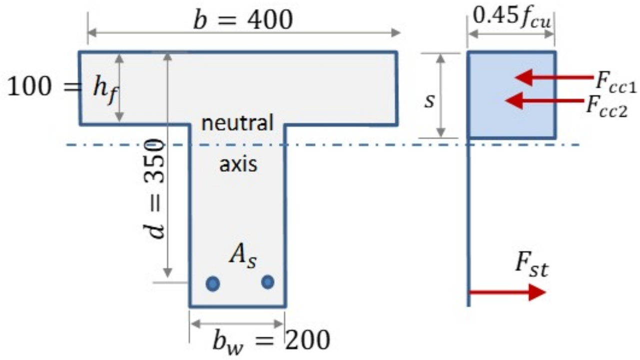

Figure 2.40: Design a flanged section for all positive moments.

Example: $M = 170\,\mathrm{kN\,m} > M_f$

The stress block extends below the flange, so $s > h_f$. For the web portion:

$$K = \frac{M - m_1}{b_w d^2 f_{cu}} = \frac{(170 - 81)\cdot 10^6}{200 \cdot 350^2 \cdot 30} = 0.121 < 0.156.$$

The lever arm for the web contribution m_2 is

$$z_2 = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) \approx 294\,\mathrm{mm} > 0.775d.$$

Hence:

$$A_{s2} = \frac{M - m_1}{0.87 f_y z_2} = \frac{(170 - 81)\cdot 10^6}{0.87 \cdot 500 \cdot 294} \approx 696\,\mathrm{mm}^2.$$

Total tension steel:

$$A_s = A_{s1} + A_{s2} = 621 + 696 = 1317\,\mathrm{mm}^2.$$

Provide:

$$3T25 \quad (A_s \approx 1472\,\mathrm{mm}^2).$$

Steel ratio:

$$\rho = \frac{A_s}{b_w h} = \frac{1472}{200 \cdot 400} \approx 1.8\% < 2.5\%.$$

Case 4: Maximum Resistance Without Compression Steel, $M_{c,\max}$

The maximum resistance of the flanged cross section *without* compression reinforcement occurs at the limiting neutral axis depth $x = x_{\max} = 0.5d$.
Solution steps

1. Lever Arm at $x_{\max}$:

$$s_{\max} = 0.9x_{\max} = 0.45d, \qquad z = d - \frac{s_{\max}}{2} = 0.775d = z_{\min}.$$

2. Moment Resistance of Wing Sections at $x_{\max}$:The moment resistance of the web contribution at maximum is

$$m_{2,\max} = F_{cc2,\max} z_2 = (0.45 f_{cu} b_w \cdot 0.45d)\,(0.775d) = 115\,\mathrm{kN\,m}.$$

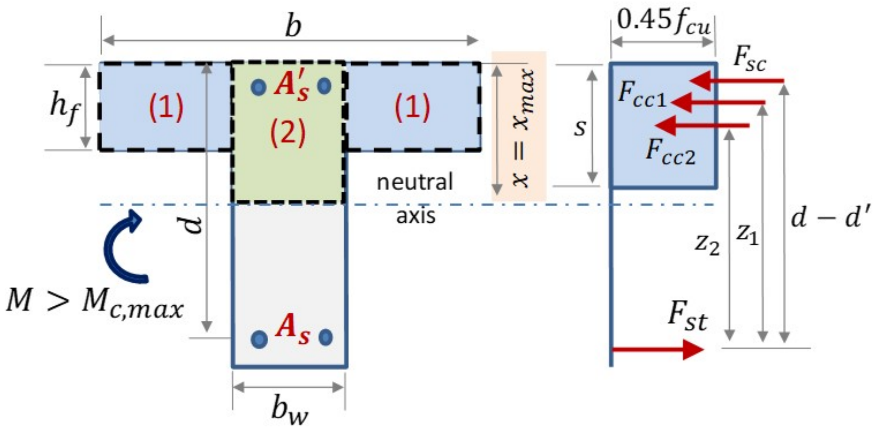

Figure 2.41: Design a flanged section for all positive moments.

3. Total maximum moment of resistance (no compression steel):

$$M_{c,\max} = m_1 + m_{2,\max} = 81 + 115 = 196\,\text{kN m}.$$

   For $M_f \le M \le M_{c,\max}$:

$$162 \le M \le 196\ \text{kN m}.$$

4. Steel Area at $M_{c,\max}$: Reinforcement area due to web resistance:

$$A_{s2} = \frac{m_{2,\max}}{0.87 f_y z_{\min}} = \frac{115 \cdot 10^6}{0.87 \cdot 500 \cdot 0.775d} \approx 978\,\text{mm}^2.$$

   Total steel area:

$$A_s = A_{s1} + A_{s2} = 621 + 978 = 1599\,\text{mm}^2 \approx 2T32\ (A_s \approx 1608\,\text{mm}^2).$$

   Steel ratio:

$$\rho = \frac{A_s}{b_w h} = \frac{1608}{200 \cdot 400} = 2.0\% < 2.5\%.$$

Case 5: $M \ge M_{c,\max}$ **(Compression Steel Required)**

For $M > M_{c,\max}$, compression reinforcement is required. At this stage:

$$x = x_{\max} = 0.5d, \qquad z = z_{\min} = 0.775d.$$

Example: $M = 215\,\text{kN m} \ge M_{c,\max}$

Solution steps

1. The additional moment $(M - M_{c,\max})$ is resisted by compression steel. Taking moments about the tension steel level, the required compression steel area is

$$A'_s = \frac{M - M_{c,\max}}{0.87 f_y (d - d')} = \frac{(215 - 196) \cdot 10^6}{0.87 \cdot 500 \cdot (350 - 50)} \approx 125\,\text{mm}^2 \approx 2T12\ (A'_s \approx 226\,\text{mm}^2).$$

2. Tension steel area is then

$$A_s = \frac{0.2 f_{cu} b_w d + 0.45 f_{cu}(b - b_w) h_f}{0.87 f_y} + A'_s = 1598 + 226 = 1824\,\text{mm}^2 \approx 4T25\ (A_s \approx 1963\,\text{mm}^2).$$

3. Steel Ratio:
$$\rho = \frac{A_s}{b_w h} = \frac{1963}{200 \cdot 400} = 2.45\% < 2.5\%.$$

Case 6: Ultimate Capacity at $\rho_{\max} = 2.5\%$

Solution steps

1. Steel Area at $\rho_{\max}$:
$$\rho_{\max} = 2.5\% \quad \Rightarrow \quad A_s = \rho_{\max} b_w h = 0.025 \cdot 200 \cdot 400 = 2000\,\mathrm{mm}^2.$$

2. Steel area due to wings:
$$A_{s1} = 621\,\mathrm{mm}^2,$$

3. so the web steel area is
$$A_{s2} = A_s - A_{s1} = 2000 - 621 = 1379\,\mathrm{mm}^2.$$

4. From force equilibrium:
$$F_{st} = F_{cc1} + F_{cc2} + F_{sc},$$
we obtain the compression steel area:
$$A'_s = A_s - \frac{0.45 f_{cu}}{0.87 f_y}\left(b_w \cdot 0.45d + (b - b_w)h_f\right) \approx 402\,\mathrm{mm}^2 \approx 2T16.$$

5. The moment equilibrui yields the ultimate moment of resistance for maximum reinforcement is
$$\begin{aligned} M_{\mathrm{ult}} &= F_{sc}(d - d') + F_{cc1}\left(d - \frac{h_f}{2}\right) + F_{cc2} z_{\min} \\ &= 0.87 f_y A'_s (d - d') + 0.45 f_{cu}(b - b_w) h_f \left(d - \frac{h_f}{2}\right) + 0.45 f_{cu} b_w \cdot 0.45d \cdot 0.775d \\ &\approx 249\,\mathrm{kN\,m} \end{aligned}$$

.

Summary: Required Reinforcement for Different Moment Ranges

M (kNm)	A_s (mm^2)	A'_s (mm^2)	$\rho = \frac{A_s}{b_w h}$
$0 \le M \le M_{\min} = 24$	$A_{s,\min} = 480$	0	$0 \le \rho \le 0.3\%$
$M_{\min} \le M \le M_f = 162$	$480 \le A_s \le 1241$	0	$0.3\% \le \rho \le 1.6\%$
$M_f \le M \le M_{c,\max} = 196$	$1241 \le A_s \le 1608$	0	$1.6\% \le \rho \le 2.0\%$
$M_{c,\max} \le M \le M_{\mathrm{ult}} = 249$	$1608 \le A_s \le 2000$	$0 \le A'_s \le 402$	$2.0\% \le \rho \le 2.5\%$

2.7 Complex Cross-Sectional Geometries

We briefly explore designing flanged sections with complex geometries, such as a double-web flanged section and a trapezoidal web section.

2.7.1 Inverted Hollow box section

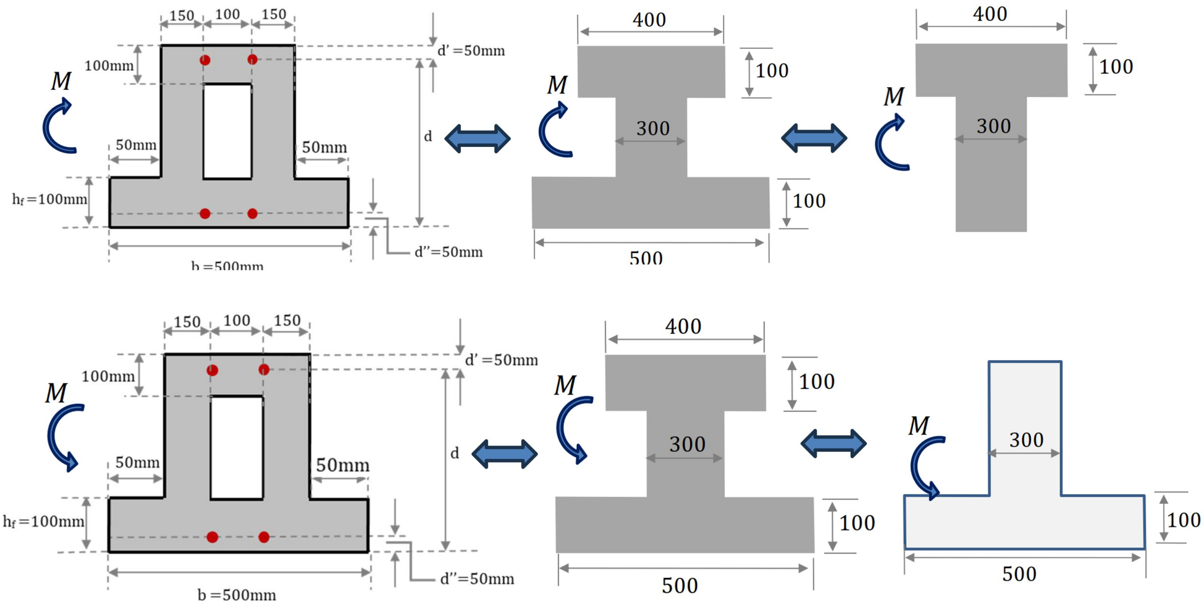

Figure 2.42: Inverted Hollow box section.

2.7.2 Trapezoidal Web Section

For a flanged section with a trapezoidal web (width varying from b_w at the top to b'_w at the bottom), the compressive force in the web changes with depth. If the neutral axis is below the flange, the wings are handled separately, and the web's compressive force is:

$$F_{cc,\text{web}} = 0.45 f_{cu} \int_0^{s-h_f} b(z)\, dz$$

where $b(z)$ varies linearly from b_w to b'_w. This requires integration, but the problem can be approximated by treating the web as a series of rectangular segments.

2.8 Axial Load and Bending Moment: Column sections

While we will study the design of columns more systematically in a later Section, herein we examine how to design an R/C section subjected to both axial load N and bending moment M, a common scenario in columns. Again, our analysis approach is hierarchical (Figure 2.43), we first examine the design problem on the level of the cross section, so we can later design columns (structural member - level) and eventually entire buildings (structural system - level). The axial force is usually compressive, which conventionally in this course we represent as a positive axial force $N > 0$.

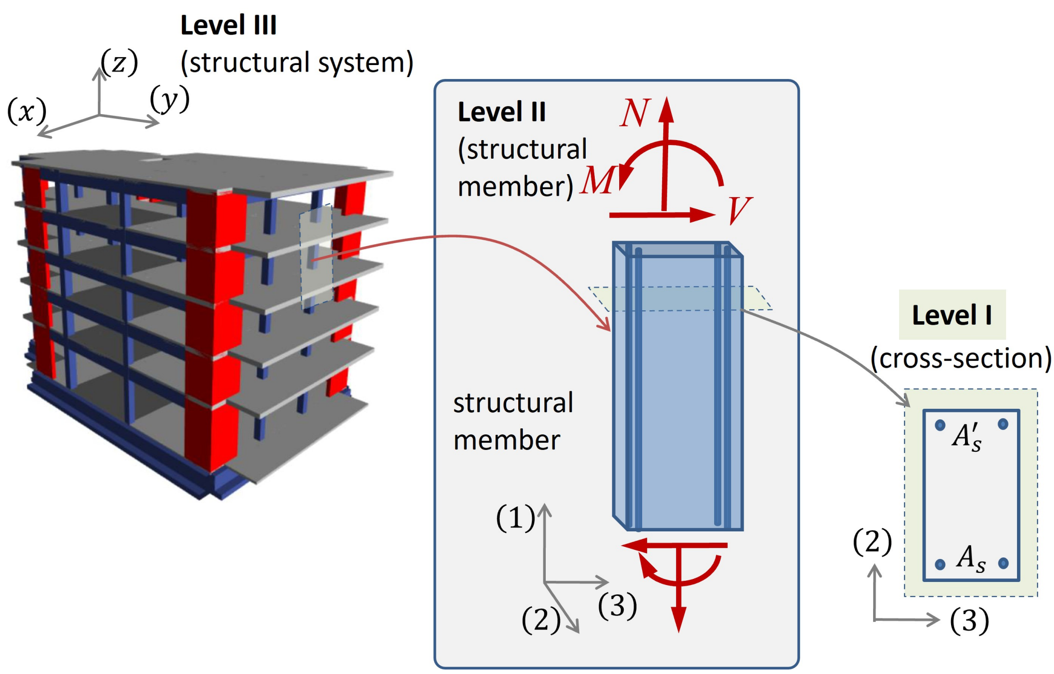

Figure 2.43: Hierarchical approach: from the structural system to the cross section of a column

Recall, that any axial load N not applied at the centroid of the cross-section is equivalent with a combination of an equal axial force N applied at the centroid, plus the moment produced by the axial force times the eccentricity e between the point of application of the force and the centroid of the cross-section $e \cdot N$ see Figure 2.44. Theoretically the combinations of axial forces N and the bending moments M are infinite, however, it is elucidating to consider two characteristic cases:

1. Dominant bending moment with small axial force (Figure 2.45 top).
2. Dominant axial force with small bending moment (Figure 2.45 bottom).

As expected, the smaller the axial load the closer the stress and strain profiles resemble the problem of a cross section of a beam, which we examined assuming pure bending conditions $M \neq 0, N = 0$. Conversely, the larger the axial load, the closer the stress and strain profiles resemble the case of pure axial compression $M = 0, N > 0$. Observe however, that as a result in the latter case the neutral axis depth x can be larger than the effective depth of the section d, and in the limit case of pure compression with zero bending moment x tends to infinity as the stresses become uniform along the whole depth of the cross section.

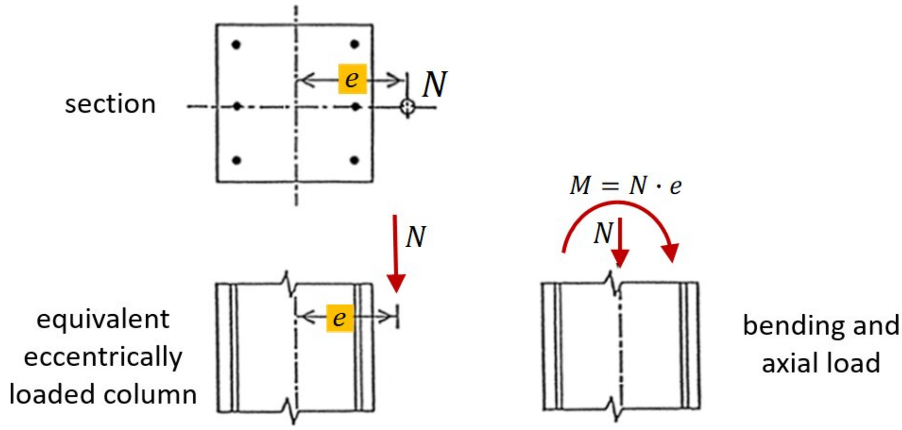

Figure 2.44: Axial load plus uniaxial bending: plan view (top left), elevation (bottom)

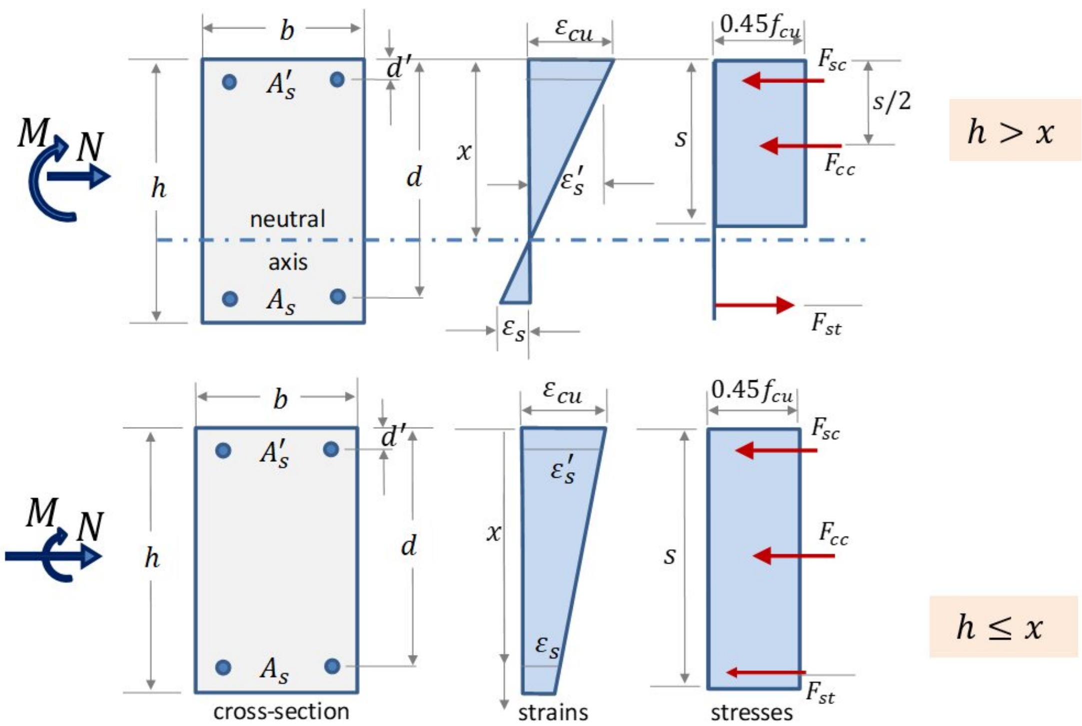

Figure 2.45: Axial load plus bending at ULS: the cross-section of a member with typical strain and stress distributions for varying positions of the NA at the ULS. Top: Dominant bending moment with small axial force. Bottom: Dominant axial force with small bending moment.

2.8.1 Design Formulae for Force and Moment Equilibrium

The following equations outline the force and moment equilibrium conditions for a reinforced concrete section under axial load N and bending moment M loading, considering both general and symmetric reinforcement arrangements. There are again two useful equilibrium conditions: one force equilibrium (along x-axis), and one moment equilibrium about any point.

Force Equilibrium (General Case)

The applied axial load N (positive for compression, negative for tension) adds an additional force to the equilibrium equation compared to the beam cross-section equilibrium. Recall that N is known from structural analysis, thus it does not add an unknown to the equilibrium equation:

$$N = F_{cc} + F_{sc} + F_{st}$$

where:

- F_{cc}: Compressive force developed in the concrete and acting through the centroid of stress block,
- F_{sc}: Compressive force in the reinforcement area A'_s and acting through its centroid,
- F_{st}: tensile or compressive force in the reinforcement area A_s: and acting through its centroid

$$N = 0.45 f_{cu} bs + f'_s A'_s + f_s A_s \tag{2.23}$$

where:

- f'_s: Compressive stress in the reinforcement A'_s,
- f_s: Tensile or compressive stress in the reinforcement A_s (f_s is negative when in tension).
- Observe that we have not assumed yielding of steel, but instead we keep the equation generic by using the steel stresses f_s and f'_s.

Moment Equilibrium (General Case)

Taking moments about the mid-depth ($h/2$) of the section, the applied moment (M) is balanced by:

$$\begin{aligned} M &= F_{cc}\left(\frac{h}{2} - \frac{s}{2}\right) + F_{sc}\left(\frac{h}{2} - d'\right) - F_{st}\left(d - \frac{h}{2}\right) \\ M &= 0.45 f_{cu} bs\left(\frac{h}{2} - \frac{s}{2}\right) + f'_s A'_s\left(\frac{h}{2} - d'\right) - f_s A_s\left(d - \frac{h}{2}\right) \end{aligned} \tag{2.24}$$

Note that the axial force N passes through the mid-depth, so it does not contribute to the moment.

Design Formulae

Note that at this point the mathematical formulation of the cross-section under combined axial load N and bending moment M is not solvable. The two equations (Equation 2.23 and Equation 2.24) involve three unknowns: A_s, A'_s and s. However, the closer the strain and stress profiles resemble the case of pure uniaxial loading, the less suitable it is to use a non symmetric reinforcement arrangement. For a symmetric arrangement of reinforcement, the areas of compression and tension reinforcement are equal:

$$A_s' = A_s = \frac{A_{sc}}{2}$$

where A_{sc} represents the total amount of vertical reinforcement. For such a symmetric reinforcement ($A_s' = A_s = A_{sc}/2$), the number of unknowns reduces to two: the total steel area A_sc, and the depth of the stress block s, and the mathematical formulation becomes solvable. The two equations (force and moment equilibrium) have to be solved simultaneously. To avoid involved calculations and keep the solution procedure practical, the design codes introduce design charts (M-N interaction diagrams in dimensionless form) such as the one depicted in Figures 2.46 and 2.47. The axes of such charts read in terms of (dimensionless) axial load ratio and (dimensionless) bending moment ratio:

$$\frac{N}{bhf_{cu}} = \text{(dimensionless) axial load ratio} \tag{2.25}$$

$$\frac{M}{bh^2 f_{cu}} = \text{(dimensionless) bending moment ratio} \tag{2.26}$$

Therefore, any combination of N and M is presented as a point on the chart whose co-

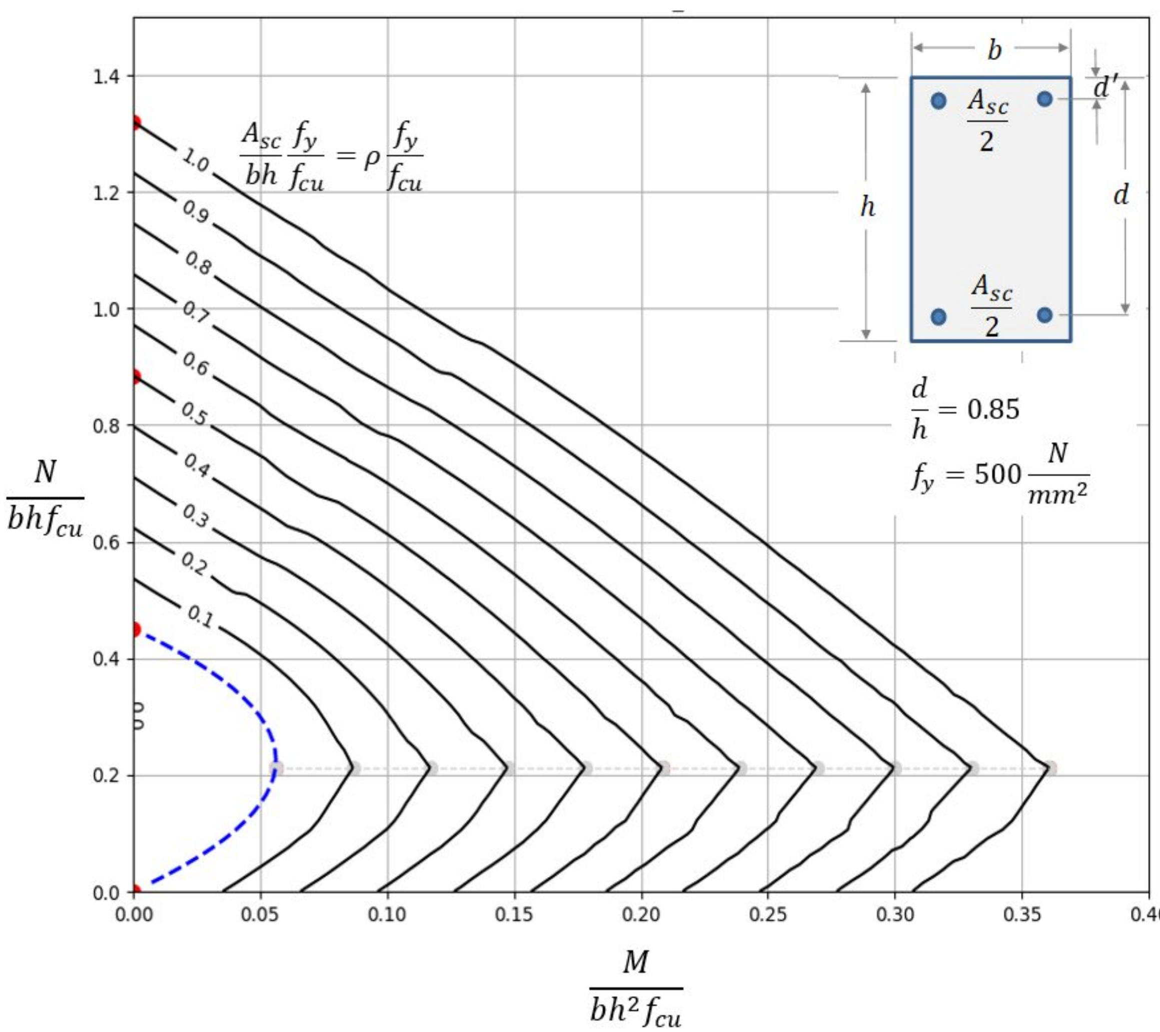

Figure 2.46: Design charts: $f_y = 500\,\mathrm{N/mm^2}$, $d/h = 0.85$

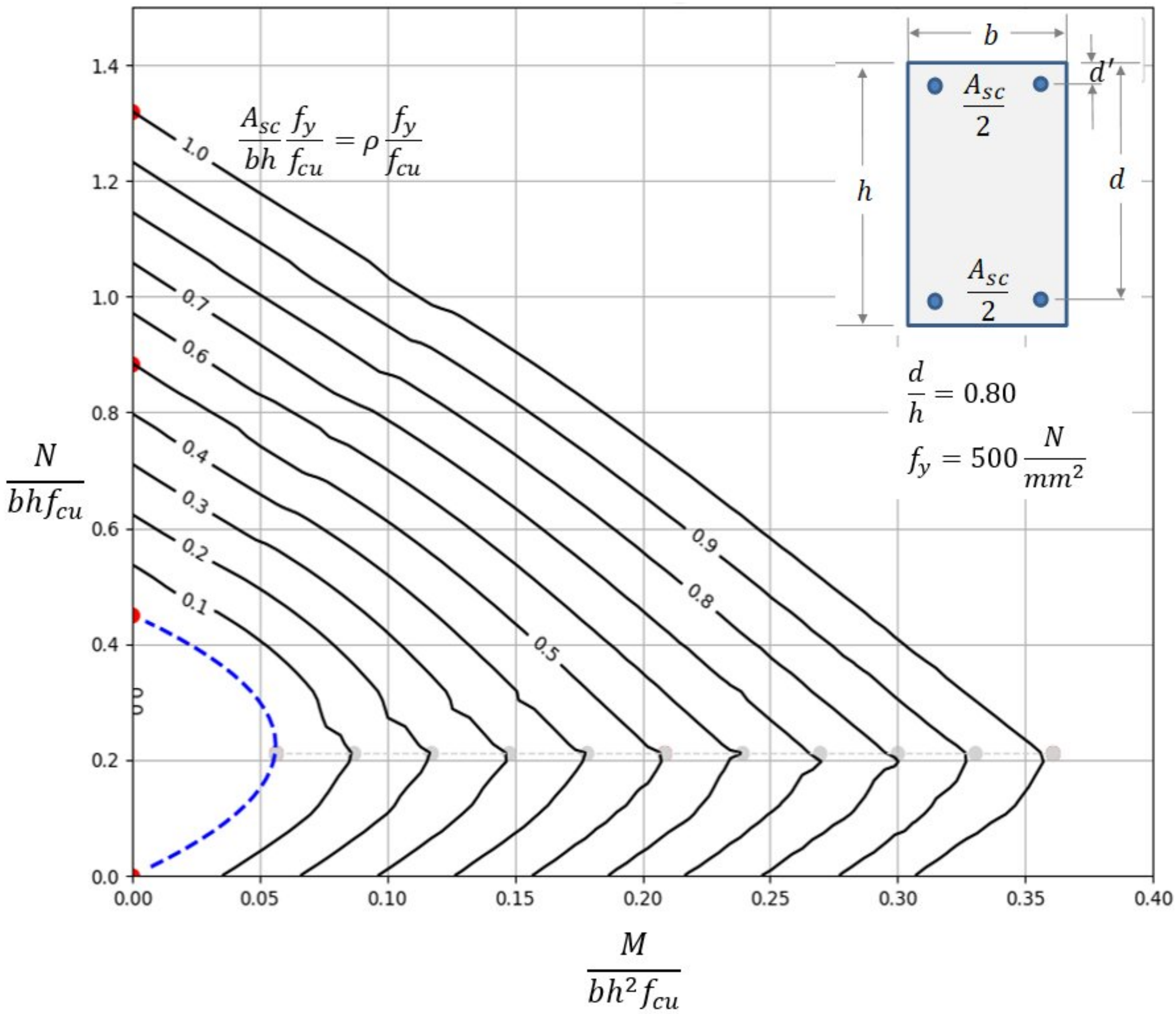

Figure 2.47: Design charts: $f_y = 500\,\mathrm{N/mm^2}$, $d/h = 0.80$

ordinates are the axial load and bending moment ratios. The reinforcement areas are represented as contours on the plane where each contour curve correspond to a different level of:

$$\frac{A_{sc} f_y}{bhf_{cu}} = \rho \frac{f_y}{f_{cu}} \tag{2.27}$$

where A_{sc} is the area of steel reinforcement, f_y is the yield strength of steel, b and h are the cross-section dimensions, and f_{cu} is the concrete compressive strength. Once the $N - M$ combination of interest is mapped to a point on the plane of the chart, we estimate the corresponding value $\frac{A_{sc} f_y}{bhf_{cu}}$ which allows to calculate the required steel area A_{sc}. Note that charts differ based on the value of the tensile strength f_y and the cross-section geometry d/h. Since it is unlikely that a chart of the exact specific geometry of interest (i.e., d/h value) is available for each case of interest, we can estimate the solution using two charts one of a lower d/h value and one of a higher d/h value and interpolate using the two solutions. Importantly, the solution does not have to be extremely accurate.

In summary, the **solutions steps** using a design chart are:

1. Calculate the *axial load ratio*: $\frac{N}{f_{cu} bh}$,
2. Calculate the *moment ratio*: $\frac{M}{f_{cu} bh^2}$, where:
   - N: Axial load,
   - M: Bending moment,
   - f_{cu}: Concrete strength,
   - b, h: Cross-section dimensions.
3. Plot the point $(\frac{N}{f_{cu} bh}, \frac{M}{f_{cu} bh^2})$ on the interaction diagram.
4. Read the reinforcement ratio from the diagram contours.

2.8.2 Example: Design of a Short Column Section

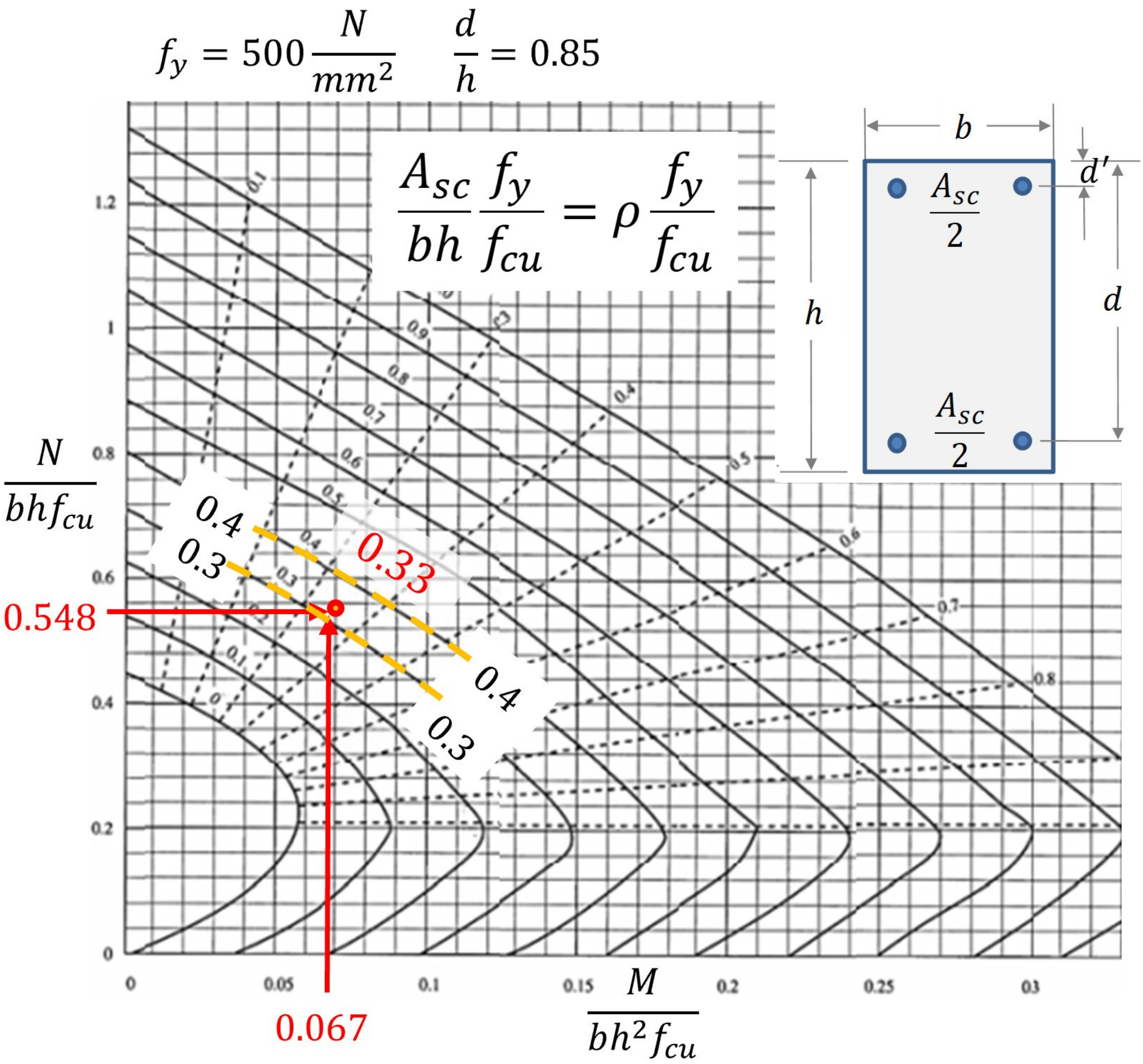

Figure 2.48: Design chart: $f_y = 500\,\mathrm{N/mm^2}$, $d/h = 0.80$

Design of a short column section to determine the total amount of steel reinforcement (A_{sc}) required.

Given Data

- Section dimensions: $300\,\mathrm{mm} \times 300\,\mathrm{mm}$, with $\frac{d}{h} = 0.85$,
- Design axial load: $N = 1480\,\mathrm{kN}$,
- Design moment: $M = 54\,\mathrm{kN\,m}$,
- Material properties: $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$.

Solution steps

1. Axial Load Ratio (see Equation 2.25):

$$\frac{N}{bhf_{cu}} = \frac{1480 \cdot 10^3}{30 \cdot 300 \cdot 300} = 0.548 \approx 0.55$$

2. Moment Ratio (see Equation 2.26):

$$\frac{M}{bh^2 f_{cu}} = \frac{54 \cdot 10^6}{30 \cdot 300 \cdot 300^2} = 0.0667 \approx 0.067$$

3. Using the design chart of Figure 2.48 $\frac{d}{h} = 0.85$, the point (0.548, 0.067) lies between contours 0.3 and 0.4, approximately at 0.33, thus, the required total steel reinforcement areas (A_{sc}) is (see Equation 2.27):

$$\frac{A_{sc} f_y}{bhf_{cu}} \approx 0.33 \rightarrow A_{sc} = 0.33 \cdot \frac{300 \cdot 300 \cdot 30}{500} = 1782\,\mathrm{mm}^2$$

   Provide 4T25 (i.e., 4 bars of $25\,\mathrm{mm}$ diameter) $A_{sc} = 1963\,\mathrm{mm}^2$) placed symmetrically (2 bars at the top, 2 at the bottom).

2.8.3 M-N Interaction Diagram

The $M - N$ interaction diagram represents the relationship between axial load (N) and bending moment (M) at failure for a reinforced concrete section, derived from:

$$N = 0.45 f_{cu} bs + f_s' A_s' + f_s A_s$$

$$M = 0.45 f_{cu} bs \left(\frac{h}{2} - \frac{s}{2} \right) + f_s' A_s' \left(\frac{h}{2} - d' \right) - f_s A_s \left(d - \frac{h}{2} \right)$$

The diagram plots N (vertical axis) against M (horizontal axis), defining a failure envelope:

- **Inside the curve** there are safe combinations of M and N.
- **Outside the curve** there are unsafe combinations that lead to failure.

Effect of Axial Load

Compression N increases moment capacity up to the balanced condition, further increase of compression beyond the balanced condition decreases the moment capacity of the cross-section. Additionally, tensile axial forces also decrease the moment capacity of the cross-section. Thus low level compression is beneficial for the resistance of the section, as opposed to high level compression or tension. For this reason, design codes prefer designs in the region below the balanced point (tension-dominated) ($N < N_{bal}$, but above the horizontal axis ($N \geq 0$) to avoid tensile axial loads.

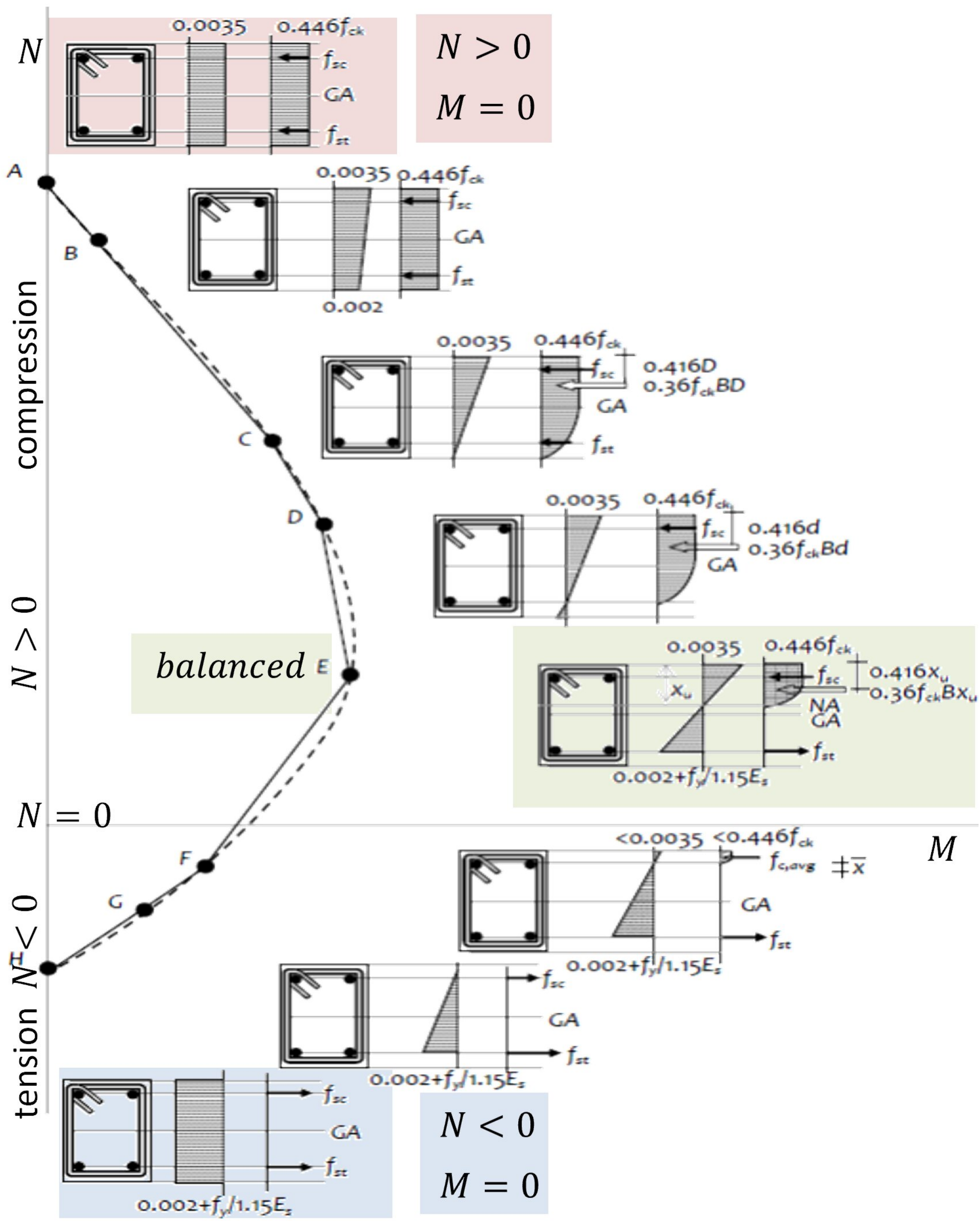

Figure 2.49: $M - N$ interaction diagram. This plot illustrates the failure envelope for a reinforced concrete section subjected to combined axial load and bending moment. The curve demonstrates the range of possible failure states, including pure compression (top of the curve), balanced failure (point of tangency where concrete crushing and steel yielding occur simultaneously), pure tension (bottom left), and all intermediate states. The accompanying strain and stress profiles in the cross section correspond to these key points, showing how the distribution of strains and stresses evolves from compression-dominated to tension-dominated failure modes.

2.8.4 Types of Failure for Axial Load and Uniaxial Bending

For sections subjected to axial load (N) and uniaxial bending (M), the relative magnitudes of N and M determine the failure mode—either tension failure or compression failure. The steel strains are defined as:

$$\varepsilon_s = \varepsilon_{cu}\frac{d-x}{x}, \quad \varepsilon_s' = \varepsilon_{cu}\frac{x-d'}{x}$$

Failure modes are determined relative to the balanced failure point (N_{bal}, M_{bal}).

Balanced Failure

Balanced failure occurs when the concrete crushes ($\varepsilon_c = \varepsilon_{cu}$) and the tension steel yields ($\varepsilon_s = \varepsilon_y$) simultaneously, corresponding to a neutral axis depth $x = x_{bal}$:

- The depth of the neutral axis at balanced failure is:

$$x_{bal} = \frac{d}{1+\varepsilon_y/\varepsilon_{cu}} = 0.617d \quad \left(f_y = 500\,\mathrm{N/mm^2},\, \varepsilon_y = 0.00217\right)$$

- The force and moment equations at balanced failure (N_{bal}, M_{bal}) are:

$$N_{bal} = 0.45 f_{cu} b(0.9x_{bal}) + f_s' A_s' - 0.87 f_y A_s$$

$$M_{bal} = 0.45 f_{cu} b(0.9x_{bal})\left(\frac{h}{2} - \frac{0.9x_{bal}}{2}\right) + f_s' A_s'\left(\frac{h}{2} - d'\right) - 0.87 f_y A_s\left(d - \frac{h}{2}\right)$$

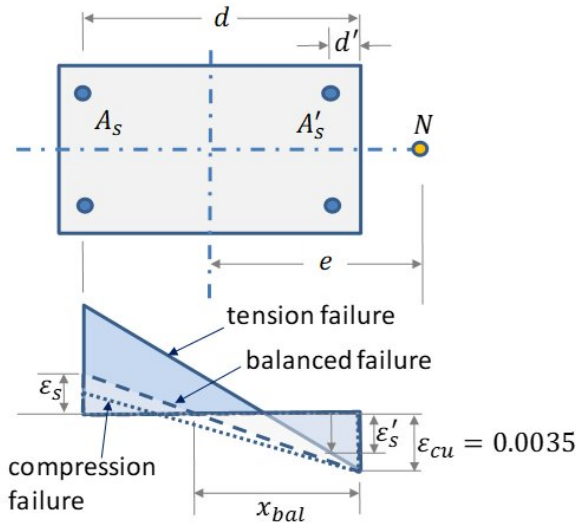

Figure 2.50: Strain distributions of different modes of failure.

Compression Failure

Compression failure occurs when the concrete crushes ($\varepsilon_c = \varepsilon_{cu}$) and the tension steel strain is less than the yield strain ($\varepsilon_s < \varepsilon_y$), corresponding to a neutral axis depth $x > x_{bal}$. This mode is associated with small eccentricity and a large depth of the neutral axis:

- Failure begins with the crushing of the concrete.

- In compression failure, crushing of the concrete occurs simultaneously with the yielding of the compression steel when:

$$x \geq \frac{d'}{1 - \varepsilon_y/\varepsilon_{cu}} = \frac{d'}{1 - 0.00217/0.0035} = 2.63d' \quad (\varepsilon_y \text{ for high-yield steel})$$

Tension Failure

Tension failure occurs when the concrete crushes ($\varepsilon_c = \varepsilon_{cu}$) and the tension steel strain exceeds the yield strain ($\varepsilon_s > \varepsilon_y$), corresponding to a neutral axis depth $x < x_{bal}$. This mode is associated with large eccentricity and a small depth of the neutral axis:

- Failure begins with the yielding of the tension steel, followed by the crushing of the concrete.

2.9 FAQs on Flexural Design of R/C Sections

This is a collection of frequently asked questions (FAQs) related to the design of reinforced concrete sections under bending and axial load, based on common student inquiries and clarifications provided during lectures.

Q1: Is axial load always handled with double reinforcement?

Yes, standard practice for columns/axial members involves reinforcement symmetrically arranged in the cross-setion.

Q2: Must the stress block depth s always be $0.9x$?

According to the HKCC 2013 code, the depth s depends on the concrete grade (f_{cu}) see Equation 2.3 and is defined as:

- $s = 0.90x$ for $f_{cu} \leq 45$ N/mm^2
- $s = 0.80x$ for $45 < f_{cu} \leq 70$ N/mm^2
- $s = 0.72x$ for $70 < f_{cu} \leq 100$ N/mm^2

Q3: How do the calculations for flanged beams differ from rectangular beams?

The main difference lies in the effective width (b_{eff}) and the neutral axis position. specifically, the design of flanged beams in flexure depends on whether the neutral axis is within the flange or in the web. We treat these two cases differently see Section **??** for more information.

Q4: Why is the flange "wing" portion of a T-beam usually considered as singly reinforced?

When $s > h_f$, we calculate the compression capacity of the wings as $0.45 f_{cu}(b - b_w)h_f$. This part is treated as a purely concrete block in compression. Only then we check if the remaining moment (carried by the web) requires compression steel.

Q5: Do the steel ratio ρ limits depend on the situation?

Not really. Minimum (ρ_{min}) and maximum (ρ_{max}) reinforcement ratios are prescribed by the HKCC2013 code of practice according to whether we design for ductility (the default case for beams and columns). We do not design for ductility in the case of Slabs.

Q6: Do we need to check both ρ_{min} and ρ_{max} in our calculations?

Yes, always check both limits to ensure the reinforcement ratio is within the allowed range. The reasons for are explained in **Section 9.9.1.2 of HKCC 2013**.

Q7: What happens if ρ exceeds ρ_{max}?

If A_s results in a $\rho > \rho_{max}$, say $2.5\,\%$, we increase section dimensions (b or h) to reduce ρ.

Q8: What is the physical significance of limiting the steel ratio to $\rho_{max} = 4\%$ and $\rho_{min} = 0.13\%$–0.3%?

See Section 2.3.7 of the Notes for a detailed explanation.

Q9: Can we ensure steel yields by simply limiting the neutral axis depth x?

Yes. By limiting $x \leq 0.5d$, we ensure the section is under-reinforced and that the steel yields before the concrete crushes. This satisfies the requirement for a ductile failure pattern, allowing for visible cracking and deflection before failure.

Q10: Are there specific steel ratio requirements for flanged beams?

Yes, the reinforcement ratio for flanged beams (e.g., T-beams) also needs to satisfy minimum and maximum limits based on the web or flange dimensions depending on the code requirement. This is covered in detail in Tutorial 05.

Q11: When do we calculate M_u?

M_u (Ultimate Moment Capacity) is calculated when evaluating an existing or previously designed section (where A_s and A_s' are already known) to see how much moment it can resist. In design, we first analyze the structure to find the required design moment M and then ensure $M_u \geq M$.

Q12: When is moment redistribution required and how does it affect K'?

Moment redistribution is used when designing continuous beams to optimize the design by shifting moments. In this course you will be told when to apply moment redistribution. The limit for K (K') changes depending on the amount of redistribution (β_b) see Table 2.6 for the effect of moment redistribution on K'.

- $K' = 0.156$ for 0% redistribution (standard limit).
- K' decreases (e.g., 0.132 for 10% redistribution) as you redistribute more moment, to ensure sufficient rotation capacity.

Q13: Do we need to check if the steel bars have yielded?

The short answer is yes. Practically though, this check is required for compression stee The common approach during design is to assume steel has yielded and after the section that has been designed (the required steel area has been calculated) to check the strain to verify that indeed steel yielded.

Q14: Why can't the "wing" section of a flanged beam be doubly reinforced?

The "wing" section only provides compression resistance and acts as a purely compressive concrete block. There is no tension force in the wing, so adding tension steel is not feasible.

Q15: Why is K not calculated for the "wing" section of a flanged beam?

$K = M/(bd^2 f_{cu})$ is a normalized moment used to find the neutral axis depth for the whole section. The "wings" (flanges) are assumed to be entirely in compression. We already know the lever arm for the wing portion ($z_1 = d - 0.5h_f$), so K is unnecessary for that part of the calculation.

Q16: Why do we check $z \geq 0.775d$?

By limiting the neutral axis depth $x \leq 0.5d$ (for $f_{cu} \leq 45$), the corresponding lever arm z is limited. Since $z = d[0.5 + \sqrt{0.25 - K/0.9}]$, the limit $x = 0.5d$ corresponds to $z_{min} = 0.775d$.

Q17: Where should compression reinforcement be placed?

Compression reinforcement (A_s') should be placed in the compression zone of the section; i.e., at the top of a beam cross-section for a sagging moment and at the bottom for a hogging moment.

Chapter 3

Shear

Overview: This section covers the following topics:

- Analysis of crack patterns in a three-point test, predicting vertical and diagonal cracks using normal and shear stresses.
- Principal stress trajectories in R/C beams, explaining crack orientations under tensile and compressive stresses.
- Shear span-to-depth ratio (a_v/d), its definition, and influence on shear failure modes (deep beam, shear-compression, shear-tension, flexural).
- Shear resistance mechanisms in R/C beams without shear reinforcement: uncracked concrete, aggregate interlock, and dowel action.
- Design concrete shear stress (v_c) calculation, quantifying shear resistance with longitudinal reinforcement.
- Shear reinforcement design using the 45° truss model, including links and bent-up bars, with practical examples and requirements.

Motivation

This Chapter discusses shear design in reinforced concrete (R/C) members. Shear design is complex due to cracking and lack of a unified theory [8, 12]. So far, we have overlooked shear forces (V), and focused on cross-sections under solely bending moment (M), or under combined bending moments (M) and axial forces (M, N). Refer again to the motivational example of Figure 1.4, from the viewpoint of stresses, we have examined solely the effect of normal stresses on cross-sections of R/C members. In this Chapter we explore how to design R/C members for shear, starting with theoretical concepts of mechanics of materials and moving to practical R/C applications.

Challenges in Shear Design

Unlike flexure (bending) where we established a consistent theoretical framework based on a small set of clear assumptions which translate into a set of design equations, shear design lacks a rigorous pertinent foundation. Instead, it relies on experimental data and observations (e.g., four-point experimental bending tests) rather than pure theory. It is partially a data-based approach.

3.1 Three-point Test and Crack Patterns

Consider a beam during a three-point test (Figure 3.1) and let's examine if we can predict the crack paths observed in such a test of an R/C beam using stresses. Recall from undergraduate mechanics of materials, that

- **Normal stresses** (σ) are perpendicular to the cross-section, and follow a linear distribution along the height of the section (Figure 3.1 right).
- **Shear stresses** (τ) are in the plane of the cross-section, and follow a parabolic distribution along the height of the section (Figure 3.1 right), zero at the top and bottom, maximum at the neutral axis (NA).

The shear stress τ of a homogeneous, elastic and uncracked beam is equal with:

$$\tau = \frac{VQ}{Ib} \tag{3.1}$$

where: V is the shear force, Q the first moment of area, I the second moment of area and b the width of the section.

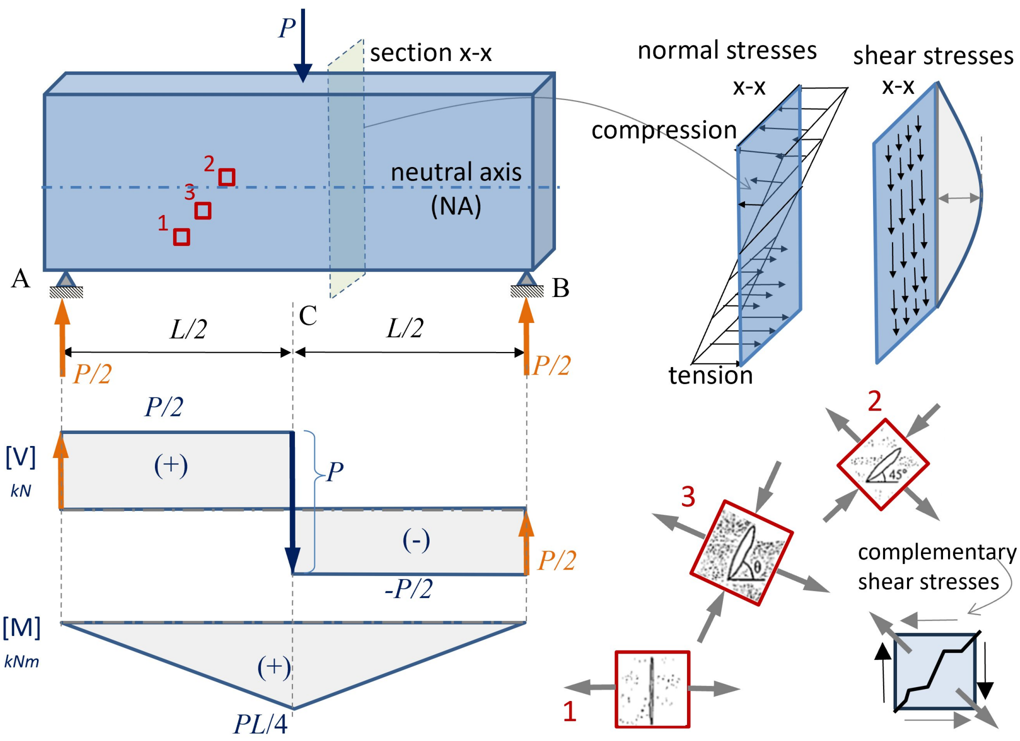

Figure 3.1: 3-point test. (top right): normal and shear stresses acting in an R/C beam, (bottom right): zoom in at 3 points at different heights in the beam and direction of resultant principal stresses (bottom right)

Consider the normal and shear stresses at three different points in the beam (Figure 3.1):

1. point 1 (at bottom): at the bottom of the beam, where the (horizontal) normal stress σ reaches its maximum tensile value $\sigma = \sigma_{max}$ and the shear stress τ is zero $\tau = 0$, we anticipate the formation of vertical cracks; named henceforth 'flexural crack'

2. point 2 (at neutral axis): Near the neutral axis (NA), where the normal stress is zero $\sigma = 0$ and shear stress τ is at its maximum $\tau = \tau_{max}$, we expect to observe 45° diagonal cracks, since the complementary shear stress gives rise to diagonal tensile and diagonal compressive stresses (Figure 3.1). Simplifying, shear failure is caused by a failure in the tension diagonal with cracks of 45°.

3. point 3 (between N.A. and bottom): Interpolating, we infer that in the region between the bottom and the neutral axis, diagonal cracks may develop at angles ranging from 45° to 90°, reflecting the effects of both combined shear and tensile stresses.

Indeed, these predictions agree with experimental observations.

Principal Stresses

More generally, crack patterns in reinforced concrete (R/C) beams arise from the complex interplay of normal and shear stresses. A natural method for analyzing these patterns involves studying the **principal stresses**, which define the directions of maximum tensile and compressive stresses within the material. Therefore, the trajectories of principal compressive stresses should indicate the paths along which cracks propagate. For instance consider the principal stresses in a simply supported beam under UDL (see Figure 3.2). Cracks typically follow the trajectories, of the principal **compressive stresses** (e.g., red curve in Figure 3.2), since in the perpendicular direction lie the trajectories of principal **tensile stresses** (e.g.., green curve in Figure 3.2). For instance, cracks often form vertically at the bottom of a beam, where horizontal tensile stresses dominate, and at a 45° angle near the neutral axis, where shear stresses become significant. However, these principal stresses trajectories assume an uncracked, homogeneous medium—an idealization that does not fully represent reinforced concrete, where cracks are prevalent. For cracked reinforced concrete, these trajectories offer merely a qualitative understanding rather than a precise quantitative model.

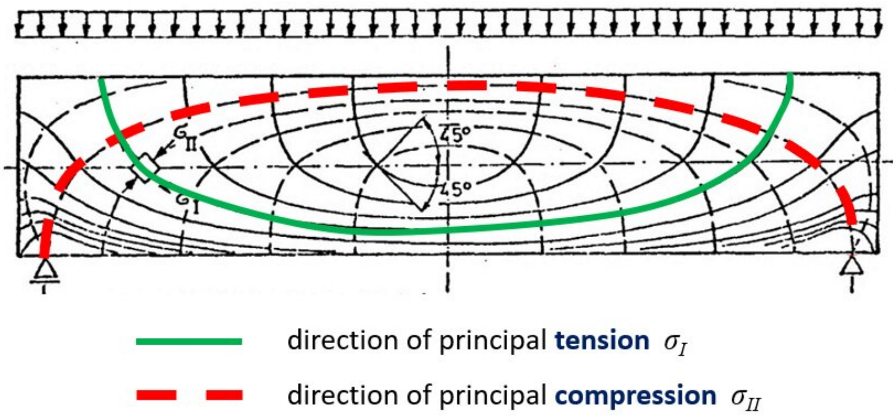

Figure 3.2: Principal stresses in an un-cracked beam

3.2 Shear Span-to-Depth Ratio (a_v/d)

Definition: Shear span (a_v) is the distance from zero moment to maximum moment. Dividing a_v by the effective depth d returns the dimensionless a_v/d ratio.

$$\frac{a_v}{d} = \frac{M}{Vd} \tag{3.2}$$

For example, for the 4-point test of Figure (1.4):

$$a_v = \frac{L}{3}, \quad \frac{a_v}{d} = \frac{M}{Vd} = \frac{PL/3}{Pd} = \frac{L}{3d}$$

For the 3-point test of Figure (3.1):

$$a_v = \frac{L}{2}, \quad \frac{a_v}{d} = \frac{M}{Vd} = \frac{PL/4}{Pd/2} = \frac{L}{2d}$$

Factors Affecting a_v/d:

- a_v/d depends on the depth (d) and the type of loading which affects the bending moment diagram, consider for example the bending moment diagram of the same beam during a 3-point vs a 4-point test.
- a_v/d is independent of the width (b) and the load level (cancels out in moment/shear relation).

3.2.1 Shear Failure Modes Based on a_v/d

Table 3.2.1 categorizes the shear failure modes depending on the value of the shear-span to depth ratio into four categories.

Table 3.1: Shear Span-to-Depth Ratio and Shear Failure Modes

Category	Shear Span/Depth Ratio	Shear Mode Failure
I	$a_v/d \leq 1$	deep beam failure
II	$1 < a_v/d \leq 2.5$	shear-compression failure
III	$2.5 < a_v/d \leq 6$	shear-tension failure (diagonal tension failure)
IV	$6 < a_v/d$	flexural failure

Category I $a_v/d < 1$ (Deep Beams): Diagonal splitting, arch action (Figure 3.3)

- Diagonal crack from support to load point, splitting the section.
- Brittle failure (**concrete crushing**) or anchorage failure of reinforcement.
- The beam behaves like an “arch” with the longitudinal reinforcement acting as the “tie.”

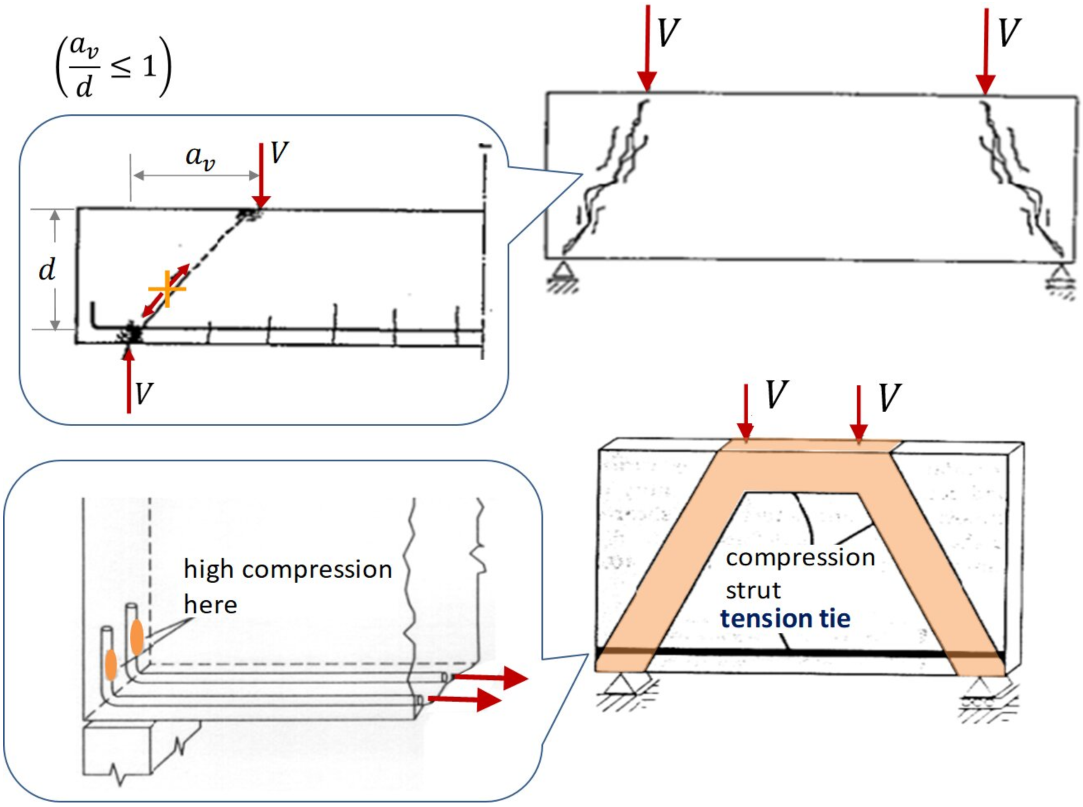

Figure 3.3: **Category I** (Deep Beams) $a_v/d < 1$: Diagonal splitting, arch action. Bottom: anchorage failure of deep beam failure.

- **Arch action**: the inclined compression zone combined with the longitudinal bottom reinforcement form a tied arch $\rightarrow$ the vertical component of the inclined zone can resist shear $\rightarrow$ diagonal compression mechanism.

Category II $1 \leq a_v/d \leq 2.5$**: Shear-compression failure, dowel failure (Figure 3.4)**

- **Shear-compression failure**: An independent diagonal crack in the web (web-shear crack) forms independently and not as a development of a flexural (vertical) crack, and eventually penetrates into the concrete compression zone at the loading point and causes concrete crushing.

- **Dowel failure**: The member may fail owing to dowel failure of the longitudinal reinforcement at the point of inclined crack.

Category III $2.5 < a_v/d \leq 6$**: Diagonal tension failure, flexural-shear cracks (Fig 3.5)**

- Flexural (vertical) cracks develope before the compressive force is great enough to develop web-shear cracks

- The flexural crack (a-b) nearest the support propagates towards the loading point, becoming an inclined crack (a-b-c) known as a **flexural-shear crack**

- Failure depends on a_v/d

- For relative low (near 2.5) a_v/d, the diagonal crack (a-b-c) spreads but stops at point j; the crack widens and propagates along the level of tension reinforcement (g-h), destroying

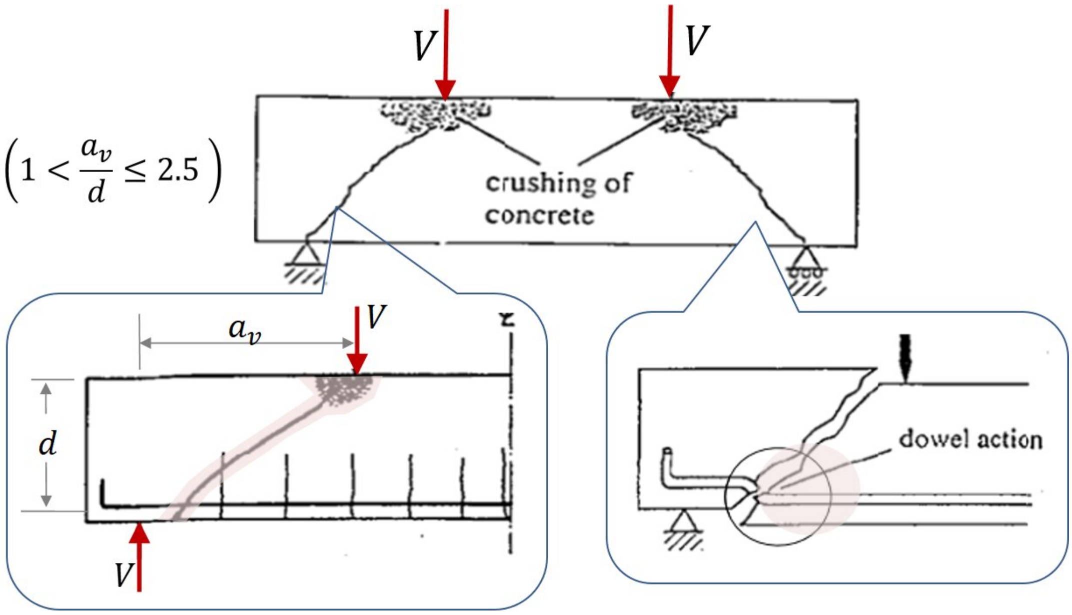

Figure 3.4: **Category II** $1 \leq a_v/d \leq 2.5$:Diagonal cracks, dowel failure

the bond between the reinforcement and the surrounding concrete; this is the **shear-bond failure** (which may also occur in Category II members)

- For relative high (near 6) a_v/d, the diagonal crack (a-b-c) spreads to e, splitting the beam into two pieces along an inclined crack (a-b-c-e)

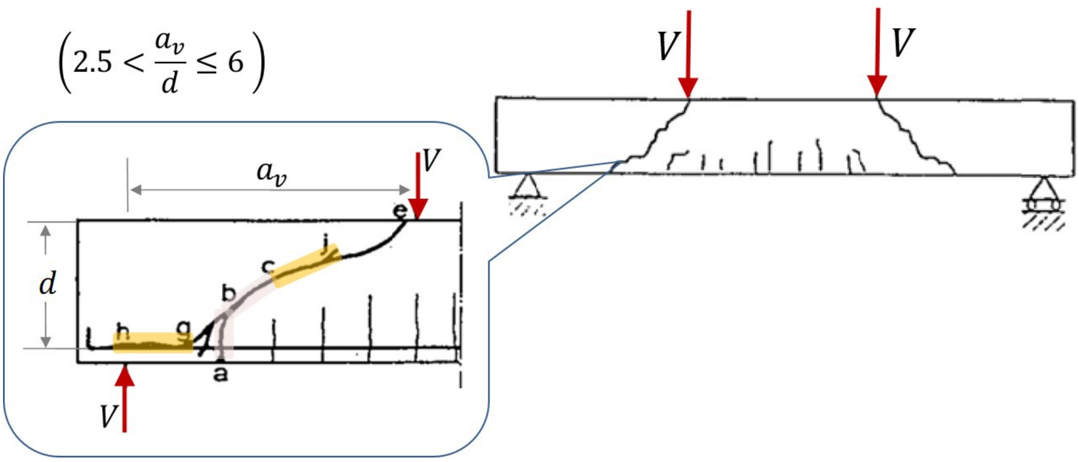

Figure 3.5: **Category III** $2.5 < a_v/d \leq 6$: Diagonal tension failure, flexural-shear cracks

Category IV $a_v/d > 6$: **Flexural failure dominates**

- Shear failure unlikely; design reverts to flexural considerations.

3.3 Shear Resistance Mechanisms (Before Shear Reinforcement)

A concrete cross-section with longitudinal steel reinforcement (but no shear reinforcement) resists shear via **three distinct mechanisms**:

1. **Uncracked Concrete (Compression Zone)**:The uncracked portion of the concrete under compression acts as a continuous medium, transferring shear. This mechanism constitutes 20–40% of shear resistance.
2. **Aggregate Interlock**: Contact forces between the two crack faces transfer shear. This is the most significant mechanism estimated to 35–50% of shear resistance.
3. **Dowel Action**: Longitudinal bars resisting forces perpendicular to their axis. This mechanism constitutes 15–25% of shear resistance.

Relative Importance: Aggregate interlock > Compression zone > Dowel action.

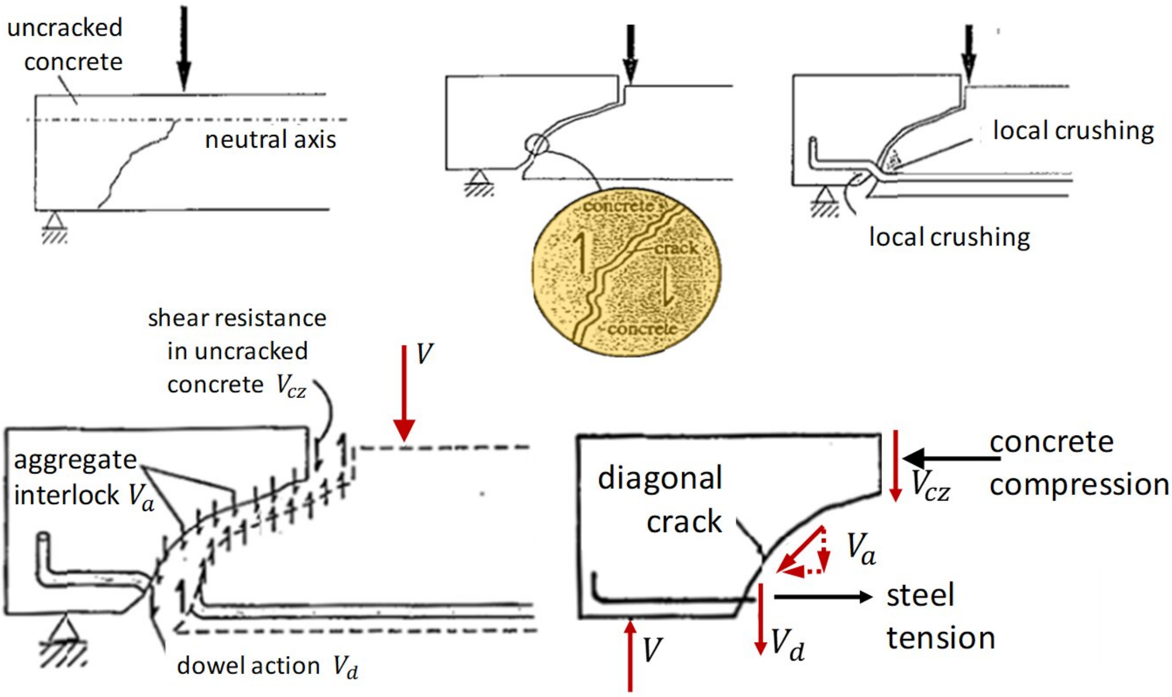

Figure 3.6: Top: shear transfer mechanisms in an R/C beam without shear reinforcement. Bottom: constituents and resultant shear forces from each of the three mechanisms of shear transfer: aggregate interlock, concrete compression, dowel action.

3.3.1 Design Concrete Shear Stress (v_c)

Definition: v_c is the design concrete shear stress, a shear stress value quantifying the shear resistance of a concrete cross section with flexural (longitudinal) reinforcement but no shear (transverse) reinforcement. It can be calculated either from the empirical equation (Equation 3.3) or equivalently obtained from a Table 3.2 (HKCC Clause 6.1.2.5(c)):

$$v_c = \frac{1}{\gamma_m}\left[0.79\left(\frac{f_{cu}}{25}\right)^{1/3}\left(\frac{100A_s}{b_v d}\right)^{1/3}\left(\frac{400}{d}\right)^{1/4}\right] \tag{3.3}$$

where:

- $\gamma_m = 1.25$: safety factor for shear (vs. 1.5 for compression)
- A_s: Area of longitudinal reinforcement (dowel action contribution)
- b_v: Shear width (web width for flanged sections).
- $b_v d$: Shear Area: to calculate an equivalent shear stress.

Equation 3.3 is an empirical formula and is subjected to the following **constraints**:

- $\frac{100A_s}{b_v d}$ should not be taken as greater than: $\frac{100A_s}{b_v d} \leq 3$
- $\frac{400}{d}$ should not be taken as less than: $\frac{400}{d} \geq 1$.
- f_{cu} should not be taken as greater than $80\,\mathrm{MPa}$ or lower than $25\,\mathrm{MPa}$.

Table 3.2: Design Concrete Shear Stress (v_c) HKCC2013 Table 6.3

$\frac{100A_s}{b_v d}$	effective depth (mm)							
	125	160	175	200	225	250	300	≥ 400
≤ 0.15	0.45	0.43	0.41	0.40	0.39	0.38	0.36	0.34
0.25	0.53	0.51	0.49	0.47	0.46	0.45	0.43	0.40
0.50	0.67	0.64	0.62	0.60	0.58	0.56	0.54	0.50
0.75	0.77	0.73	0.71	0.68	0.66	0.65	0.62	0.57
1.00	0.84	0.81	0.78	0.75	0.73	0.71	0.68	0.63
1.50	0.97	0.92	0.89	0.86	0.83	0.81	0.78	0.72
2.00	1.06	1.02	0.98	0.95	0.92	0.89	0.86	0.80
≥ 3.00	1.22	1.16	1.12	1.08	1.05	1.02	0.98	0.91

3.4 Shear Reinforcement in Reinforced Concrete Beams

The design of shear reinforcement, such as links (stirrups) and bent-up bars, relies on empirical approximations, such as the 45° truss model, supported by experimental data rather than rigorous theoretical models.

3.4.1 Types and Purpose of Shear Reinforcement

Shear reinforcement enhances the shear resistance of R/C beams by adding an additional shear resistance mechanism. The primary types of shear reinforcement are:

- **Links (Stirrups):** Vertical bars that cross diagonal shear cracks, providing the most common form of shear reinforcement (Figure 3.7).
- **Bent-up Bars:** Flexural bars bent to intersect shear cracks, offering additional resistance.

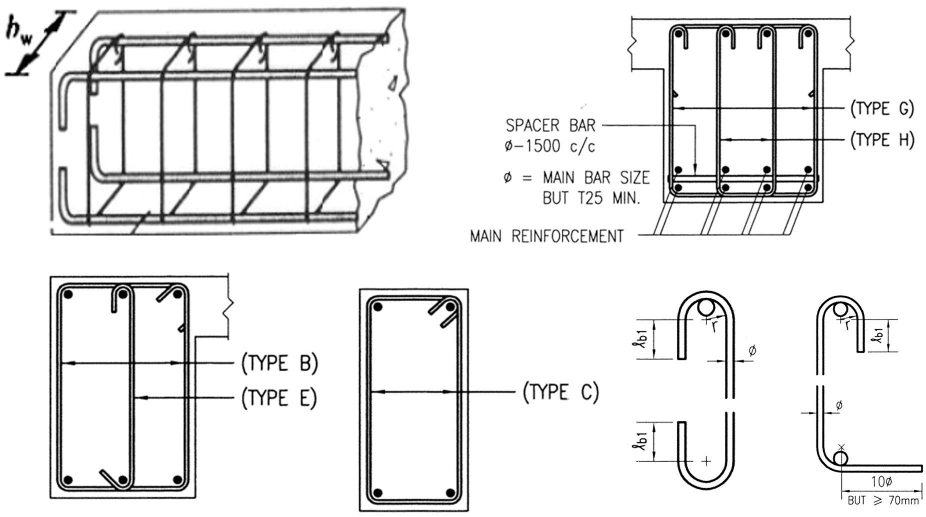

Figure 3.7: Shear reinforcement in a reinforced concrete beam.

The primary design goal is to ensure that shear reinforcement is provided so to bridge shear cracks and enhance the shear resistance of the concrete section with flexural reinforcement V_c, preventing sudden failure.

3.4.2 Shear Resistance with Links – the 45° Truss Model

The design of shear reinforcement in concrete beams is based on the concept of an analogous truss. The premise of the model is that after cracking, the beam behaves like a truss, comprised of compression and tensile members (see Figure 3.8). The analysis of trusses, with fundamental principles of statics, offers valuable insight into the behavior of truss members, such as identifying which members are in compression and which are in tension (see Appendix A.2). The compressive members are diagonal concrete struts (in between cracks), and longitudinal reinforcement under compression. The tensile members are vertical shear links and longitudinal reinforcement under tension (at the bottom of the beam for a sagging moment).

The **basic assumption** is that shear cracks form at a 45-degree angle. This 45° constant angle is an approximation, aligning with experimental observations of cracking near supports, which drastically simplifies the design model.

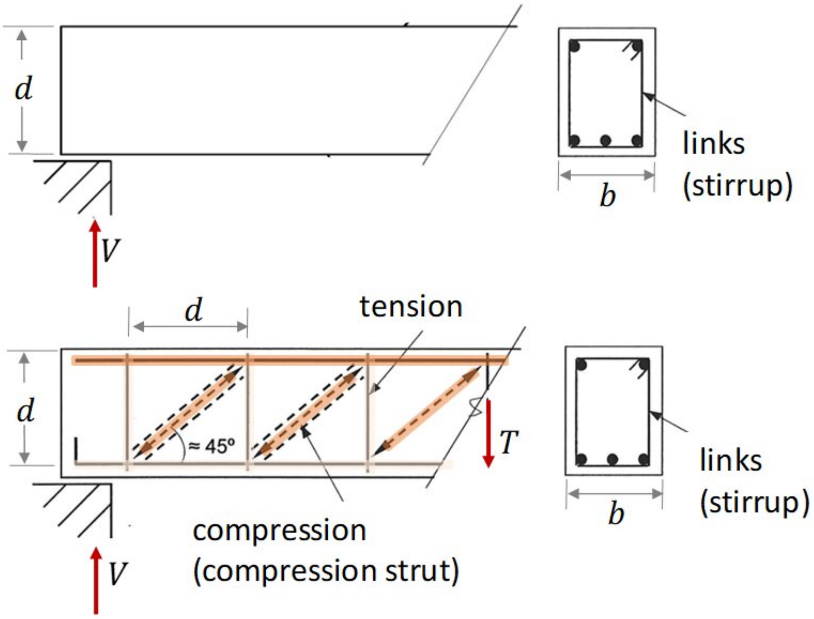

Figure 3.8: R/C beam under shear (top) and the 45° truss model for shear resistance (bottom).

The total shear resistance V of a cross-section with vertical links is the sum of the shear resistance by the shear reinforcement V_s, and the shear resistance (in terms of force) provided by the concrete V_c:

$$V = V_c + V_s \tag{3.4}$$

where introducing the corresponding design concrete shear stress v_c, with b being the beam width, and d the effective depth.

$$V_c = v_c b d \tag{3.5}$$

To calculate the shear resistance from the links V_s, we calculate first the resistance from one link, and then estimate how many links we should consider per length. The shear resistance from a single link is simply the maximum force the link can develop

$$0.87 f_{yv} A_{sv}$$

where,A_{sv} is the cross-sectional area of all legs of the link and f_{yv} is the yield strength of shear reinforcement. Adopting the assumption that the crack is propagating as at a 45° then the horizontal projection of the crack is equal with the effective depth of d. Accordingly, for links spaced at a regular distance of s_v, with $s_v < d$, the number of links crossing one crack is approximately d/s_v. Consequently, the shear resistance from the shear links is the resistance per link times the number of links crossing each crack:

$$V_s = \frac{0.87 f_{yv} A_{sv} d}{s_v} \tag{3.6}$$

Therefore, the total shear resistance of across-section reinforced with vertical links is

$$V = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right) 0.87 f_{yv} d$$

or, in terms of the shear stress $v = V_d/(b_v d)$

$$v b_v d = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right) 0.87 f_{yv} d \tag{3.7}$$

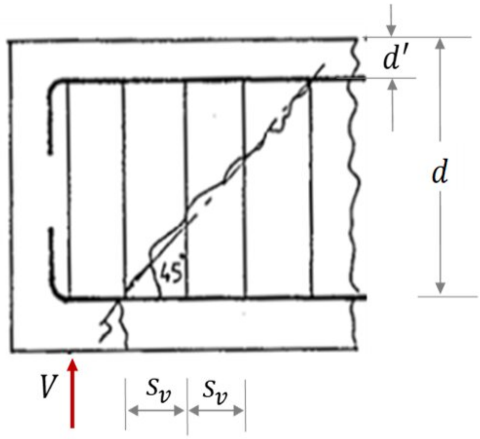

Figure 3.9: Shear link crossing a 45° diagonal crack.

Rearranging for the link-area-to-spacing-ratio, or simply, **link ratio**:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_v}{0.87 f_{yv}} \tag{3.8}$$

The link ratio is the basic design parameter; once we estimate it we can choose a link of preferred diameter, number of legs, and spacing.

Shear Resistance Calculation Example To elucidate, consider a shear link with a diameter $\phi_v = 8\text{mm}$ and a spacing $s_v = 125\,\text{mm}$:

- Area of one leg of the link: $A = \frac{\pi \phi_v^2}{4} = \frac{\pi \cdot 8^2}{4} = 50.27\,\text{mm}^2$.
- For a link/stirrup with two legs (Figure 3.10): $A_{sv} = 2 \cdot \frac{\pi \phi_v^2}{4} = 2 \cdot 50.27 = 100.53\,\text{mm}^2$.
- Link ratio: $\frac{A_{sv}}{s_v} = \frac{100.53}{125} = 0.804\,\text{mm}$.
- Typical link diameters ϕ_v range from $8\,\text{mm}$ to $16\,\text{mm}$, often made of mild steel $f_{yv} = 250\,\text{MPa}$ in regions like Hong Kong to minimize bend radii.

Table 3.3 offers precomputed values of link ratios for various stirrup diameters and spacings.

Table 3.3: A_{sv}/s_v (mm) for shear links.

Stirrup diameter (mm)	Stirrup spacing (mm)										
	85	90	100	125	150	175	200	225	250	275	300
8	1.183	1.118	1.006	0.805	0.671	0.575	0.503	0.447	0.402	0.366	0.335
10	1.847	1.744	1.57	1.256	1.047	0.897	0.785	0.698	0.628	0.571	0.523
12	2.659	2.511	2.26	1.808	1.507	1.291	1.13	1.004	0.904	0.822	0.753
16	4.729	4.467	4.02	3.216	2.68	2.297	2.01	1.787	1.608	1.462	1.34

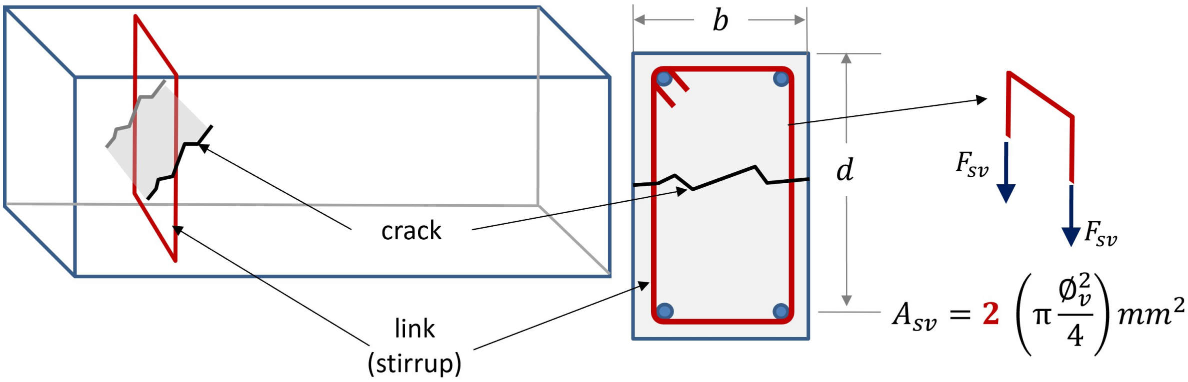

Figure 3.10: Shear link legs crossing a 45° diagonal crack.

3.4.3 Enhanced Shear Strength of Sections Close to Supports

Shear failure in beam sections without shear reinforcement typically occurs on a plane inclined at approximately 30° to the horizontal. If the failure plane is forced to incline more steeply—due to the applied load, or because the section under consideration (e.g., section X-X in Figure 3.11) is near a support—the shear force required to induce failure increases. Consequently, these sections exhibit enhanced shear resistance, while the shear force causing failure is usually that acting on section X-X.

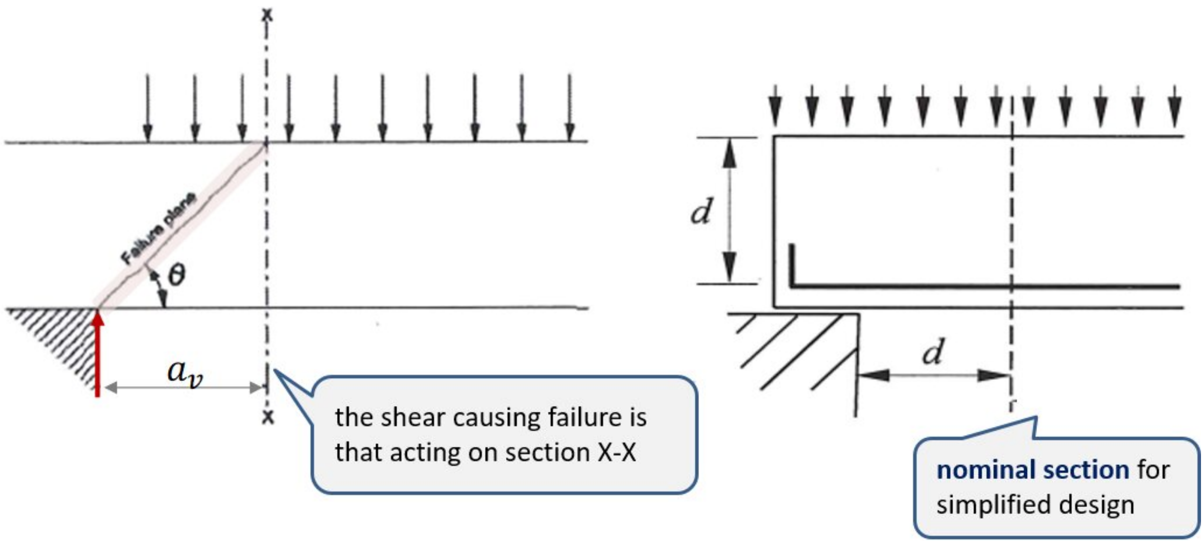

Figure 3.11: Left: Shear failure plane near support. Right: Nominal section considered for shear design is at distance d from the support.

a) Within a distance of $2d$ from the face of a support or a concentrated load, the design concrete shear stress may be increased from v_c to $2d/a_v\, v_c$, provided that the shear stress v at the face of the support remains below $\min\{0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}\}$.

b) This enhanced shear strength can be utilized in the design of sections, particularly for beams with concentrated loads near supports, or in elements such as corbels and pile caps.

c) One possible explanation for this behavior is as follows: Sections close to supports exhibit enhanced shear resistance partly due to diagonal compressive stresses induced in the concrete by the concentrated reaction. These stresses help resist shear, especially when the shear span ratio is small. For design purposes, a simplified approach (Clause 6.1.2.5(i)) is adopted to account for this enhanced shear strength.

d) A **nominal section located at a distance** d from the face of the support is typically considered, as illustrated in Figure 3.11.

- In this course, the shear stress in the nominal section will be used for the design of the shear reinforcement instead of the maximum shear stress in the face of the support.

3.4.4 Design Requirements

Table 3.4: Shear Reinforcement Requirements

Value of v **(**$\mathrm{N/mm^2}$**)**	**Form of shear reinforcement to be provided**	**Shear reinforcement to be provided**
Less than $0.5v_c$ throughout the beam	See NOTE 1	–
$0.5v_c < v \leq (v_c + v_t)$ (See NOTE 2)	Minimum links for whole length of beam	$\frac{A_{sv}}{s_v} \geq \frac{v_r b_v}{0.87 f_{yv}}$ (See NOTE 2)
$(v_c + v_t) < v < 0.8\sqrt{f_{cu}}$ or $v > 7\,\mathrm{N/mm^2}$	Links or links combined with bent-up bars. Not more than $50\,\%$ of the shear resistance provided by the steel may be in the form of bent-up bars. (See NOTE 3)	Where links only provided: $\frac{A_{sv}}{s_v} \geq \frac{(v-v_c) b_v}{0.87 f_{yv}}$ Where links and bent-up bears provided: see 6.1.2.5(e)

Notes on Table 3.4.4

- NOTE 1: Minimum links should be provided in all beams of structural importance (for this course, all beams are considered structurally important). Minimum links may be omitted in minor elements (e.g., lintels) if $v < 0.5v_c$.
- NOTE 2: Minimum links should provide a design shear stress of v_r
- NOTE 3: Provides guidance on spacing and bent-up bars (see Clause 6.1.2.5(d)).

Design Requirements for Minimum Links

When diagonal cracking occurs, the web reinforcement experiences a sudden stress increase. If the reinforcement is insufficient, this stress spike may cause immediate yielding, leading to brittle failure. To prevent sudden failure of the reinforcement, we should ensure adequate shear resistance, by providing minimum links. As per HKCC2013 [7]: Clause 6.1.2.5 / 9.2.2, the minimum links should provide a design shear resistance of at least $0.40\,\mathrm{N/mm^2}$

$$v_r = 0.4\,\mathrm{N/mm^2} \quad \text{for} \quad f_{cu} \leq 40\,\mathrm{N/mm^2}$$

adjusted for higher-strength concrete:

$$v_{\mathrm{r}} = 0.4\left(\frac{f_{cu}}{40}\right)^{2/3} \quad \text{for} \quad 40\,\mathrm{N/mm^2} < f_{cu} \leq 80\,\mathrm{N/mm^2}$$

Therefore, the minimum link ratio is:

$$\left(\frac{A_{sv}}{s_v}\right)_{\min} = \frac{0.4b_v}{0.87f_{yv}} \tag{3.9}$$

Maximum Shear Stress [HKCC2013: Clause 6.1.2.5]

This upper limit prevents crushing of the concrete in the direction of the maximum principal compressive stresses, i.e., crushing of the compression strut of the concrete. $7\,\mathrm{N/mm^2}$ is considered as the maximum design value of the shear force that can be sustained by the member. Accordingly, the design shear stress $v = \frac{V_d}{bd}$ should always satisfy:

$$v = \frac{V}{b_v d} \leq v_{max} = \min(0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}) \tag{3.10}$$

whatever shear reinforcement is provided (an allowance for $\gamma_m = 1.25$ has been included). If the design shear stress $v > v_{max}$ dimensions of the cross-section should be increased.

Maximum Spacing of Links [HKCC2013: Clause 9.2.2]

To ensure at least one stirrup intercepts one diagonal crack (according to the 45° truss model), the maximum spacing in the direction of span, $s_{v,max}$ should not be larger than

$$s_v \leq 0.75d \tag{3.11}$$

3.5 Design Examples

3.5.1 Design Example: Shear Links

Problem: Design the beam shown for shear. Simply supported beam with a span of $L = 6$ m (center-to-center of supports).

- Design (1.4G + 1.6Q) Uniformly distributed load (UDL) $q = 75.2\,\mathrm{kN/m}$.
- Longitudinal reinforcement: 2 bars of 25 mm diameter at the bottom (tension steel).
- Beam dimensions: $b = 300\,\mathrm{mm}$ (width) × $d = 550\,\mathrm{mm}$ (effective depth).
- Material grades: concrete strength: $f_{cu} = 40\,\mathrm{MPa}$, shear links: $f_{yv} = 250\,\mathrm{MPa}$, longitudinal reinforcement: $f_y = 500\,\mathrm{MPa}$.
- Support width: $300\,\mathrm{mm}$.

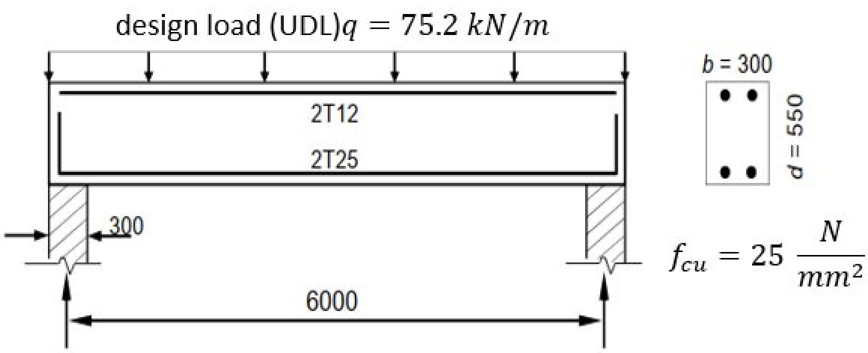

Figure 3.12: Simply supported beam - Shear Design

Solution:

Calculate the demand Shear Forces

Shear force on the center of the support (Figure 3.13):

$$V_d = \frac{qL}{2} = \frac{75.2 \cdot 6}{2} = 225.6\,\mathrm{kN}$$

Shear force on the face of the support (distance from support center = $0.3/2 = 0.15\,\mathrm{m}$) (Equation A.2):

$$V_s = V_c - q \cdot \frac{0.3}{2} = 225.6 - 75.2 \cdot \frac{0.3}{2} = 214.3\,\mathrm{kN}$$

Shear force in the nominal section ($d = 0.55\,\mathrm{m}$ of the support, Figure 3.13) (Equation A.2):

$$V_d = 214.3 - 75.2d = 214.3 - 75.2 \cdot 0.55 = 172.9\,\mathrm{kN}$$

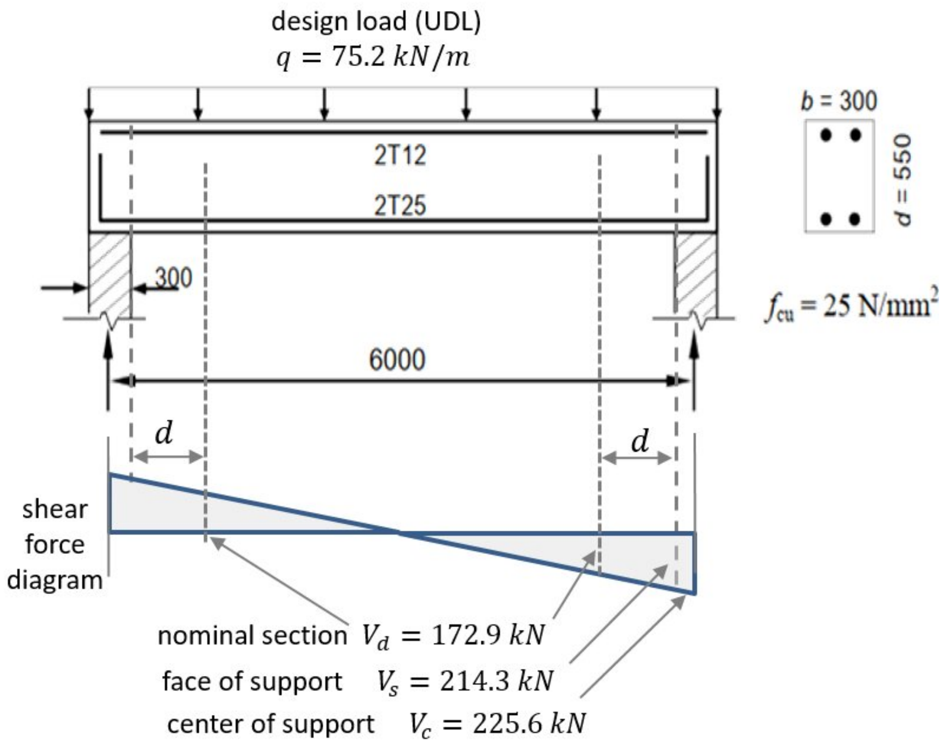

Figure 3.13: Shear Force Diagram

Maximum Shear Stress Check

First we ensure the cross-section is admissible. The maximum shear force appears at the face of the support: $V_s = 214.3\,\mathrm{kN}$. Corresponding maximum shear stress in the beam:

$$v = \frac{V}{b_v d} = \frac{214.3 \cdot 10^3}{300 \cdot 550} = 1.30\,\mathrm{N/mm^2}$$

Maximum allowable stress check:

$$v_{max} = \min(0.8\sqrt{f_{cu}}, 7) = \min(0.8\sqrt{25}, 7) = \min(4, 7) = 4.0\,\mathrm{N/mm^2} > 1.3\,\mathrm{N/mm^2} \quad \text{(OK)}$$

Since, $1.30 < 4.06\,\mathrm{N/mm^2}$ the cross section is admissible and we can proceed with the design.

Minimum Shear Links

The minimum amount of shear links required corresponds to a shear stress demand of $0.4\,\mathrm{N/mm^2}$:

$$\left(\frac{A_{sv}}{s_v}\right)_{\text{min}} = \frac{0.4 b_v}{0.87 f_{yv}} = \frac{0.4 \cdot 300}{0.87 \cdot 250} = 0.552$$

Note that we use mild steel ($f_{yv} = 250\,\mathrm{N/mm^2}$) for the shear links. From Table 3.3: for R10 links: $s_v = \frac{157}{0.552} = 284\,\mathrm{mm}$, hence provide R10@275 mm. Therefore, we choose as minimum shear links R10@275: $10\,\mathrm{mm}$ diameter shear links at $275\,\mathrm{mm}$ spacing $\frac{A_{sv}}{s_v} = 0.571$, which is sufficient.

$$\frac{A_{sv}}{s_v} = \frac{0.4 \cdot 300}{0.87 \cdot 250} = 0.552\,\mathrm{mm^2/mm}$$

These minimum shear links (R10@275) can cover a shear force demand up to:

$$V_n = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right) 0.87 f_{yd} d =$$
$$= (0.52 \cdot 300 \cdot 550 + 0.571 \cdot 0.87 \cdot 250 \cdot 550) \cdot 10^{-3} = 154.1\,\mathrm{kN}$$

Concrete Shear Stress (v_c)

The Concrete Shear Stress (v_c) provides the shear resistance of the concrete section with longitudinal steel before adding shear reinforcement. The longitudinal reinforcement under tension (we do not account for longitudinal reinforcement under compression) is (Figure 3.13) $2T25(A_s = 982\,\mathrm{mm}^2)$ hence:

$$\frac{100A_s}{b_v d} = \frac{100 \cdot 982}{300 \cdot 550} = 0.59$$

From HKCC2013 Table 6.3 or equivalently the empirical equation, the design concrete

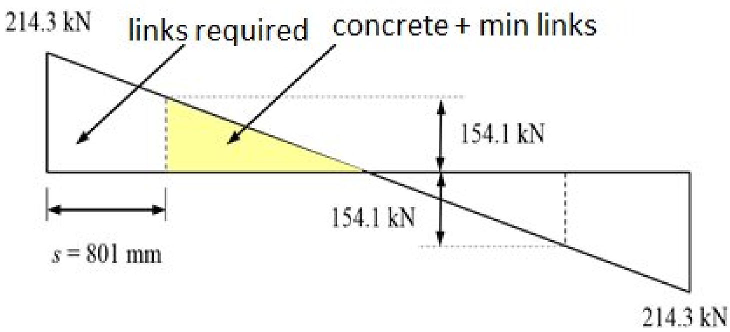

Figure 3.14: Minimum Shear Reinforcement Zone

shear stress:

$$v_c = 0.79 \left(\frac{f_{cu}}{25}\right)^{\frac{1}{3}} \left(\frac{100A_s}{b_w d}\right)^{\frac{1}{3}} \left(\frac{400}{d}\right)^{\frac{1}{4}} \frac{1}{\gamma_m} = \frac{0.79}{1.25} \left(\frac{25}{25}\right)^{\frac{1}{3}} (0.59)^{\frac{1}{3}} (1)^{\frac{1}{4}} \rightarrow$$

$v_c = 0.53\,\mathrm{N/mm^2}$.

Shear Reinforcement Needed

The shear stress in the nominal section is:

$$v = \frac{V_d}{b_v d} = \frac{172.9 \cdot 10^3}{300 \cdot 550} = 1.05\,\mathrm{N/mm^2}$$

Check if the shear stress resistance provided by the concrete design shear stress v_c, plus the minimum shear reinforcement (in terms of shear stress v_r) can cover this shear stress demand:

$$v_c + v_r = 0.52 + 0.4 = 0.92\,\mathrm{N/mm^2} < v = 1.05\,\mathrm{N/mm^2}$$

Since the answer is negative, shear reinforcement more than the minimum is required to cover the difference between the demand at the nominal section and v_c:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b}{0.87 f_{yv}} = \frac{(1.05 - 0.53) \cdot 300}{0.87 \cdot 250} = 0.731\,\mathrm{mm^2/mm}$$

Take shear links R10@200 mm (spacing rounded down from 215 mm for practicality), for R10 ($A_{sv} = 157\,\mathrm{mm}^2$) at $200\,\mathrm{mm}$ spacing:

$$\frac{A_{sv}}{s_v} = \frac{157}{200} = 0.785\,\mathrm{mm}$$

and spacing distance check

$$s_v = 200\,\mathrm{mm} < 0.75d = 413\,\mathrm{mm} \quad \text{(OK)}$$

To save reinforcement steel and cost, it makes sense to use the minimum shear links R10@275 mm for shear forces up to $V \leq 154\,\mathrm{kN}$, and only use R10@200 mm for regions of the beam where the shear force is higher than $V > 154\,\mathrm{kN}$. To calculate the regions of the beam with $V \leq 154\,\mathrm{kN}$ and $V > 154\,\mathrm{kN}$ respectively, we refer to the shear force diagram. The calculation is straightforward (we use again Equation A.2 of Appendix):

$$s = \frac{V_s}{V_n} = \frac{214.3 - 154.1}{75.2} \cdot 1000 = 801\,\mathrm{mm}$$

Concluding, we provide R10@200 mm near the supports (up to a distance of $801\,\mathrm{mm}$ from the face of the support with the first stirrup provided after $50\,\mathrm{mm}$ from the face of the support), and R10@275 mm in the intermediate region of the beam (where $V \leq 154\,\mathrm{kN}$).

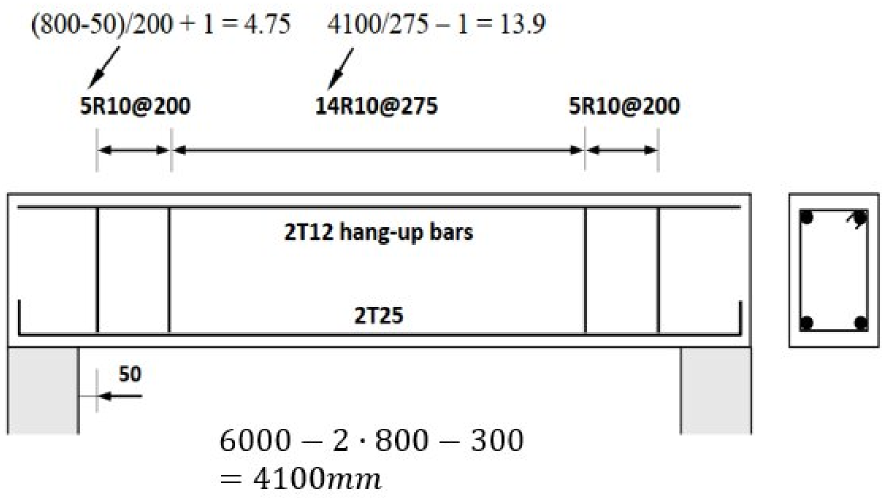

Figure 3.15: Simply supported beam - Shear Design

Additional Notes

- Nominal Section: Design shear reinforcement at a distance d from the face of the support, not at the center, to account for realistic stress distribution.
- Industry Practice: In industry shear design often relies on the shear at the face of the support (more conservative, e.g., using $214\,\mathrm{kN}$ instead of $173\,\mathrm{kN}$).
- Rounding Spacings: In practice, round spacings to multiples of 25 mm or 50 mm (e.g., 175 mm might be rounded to 150 mm).

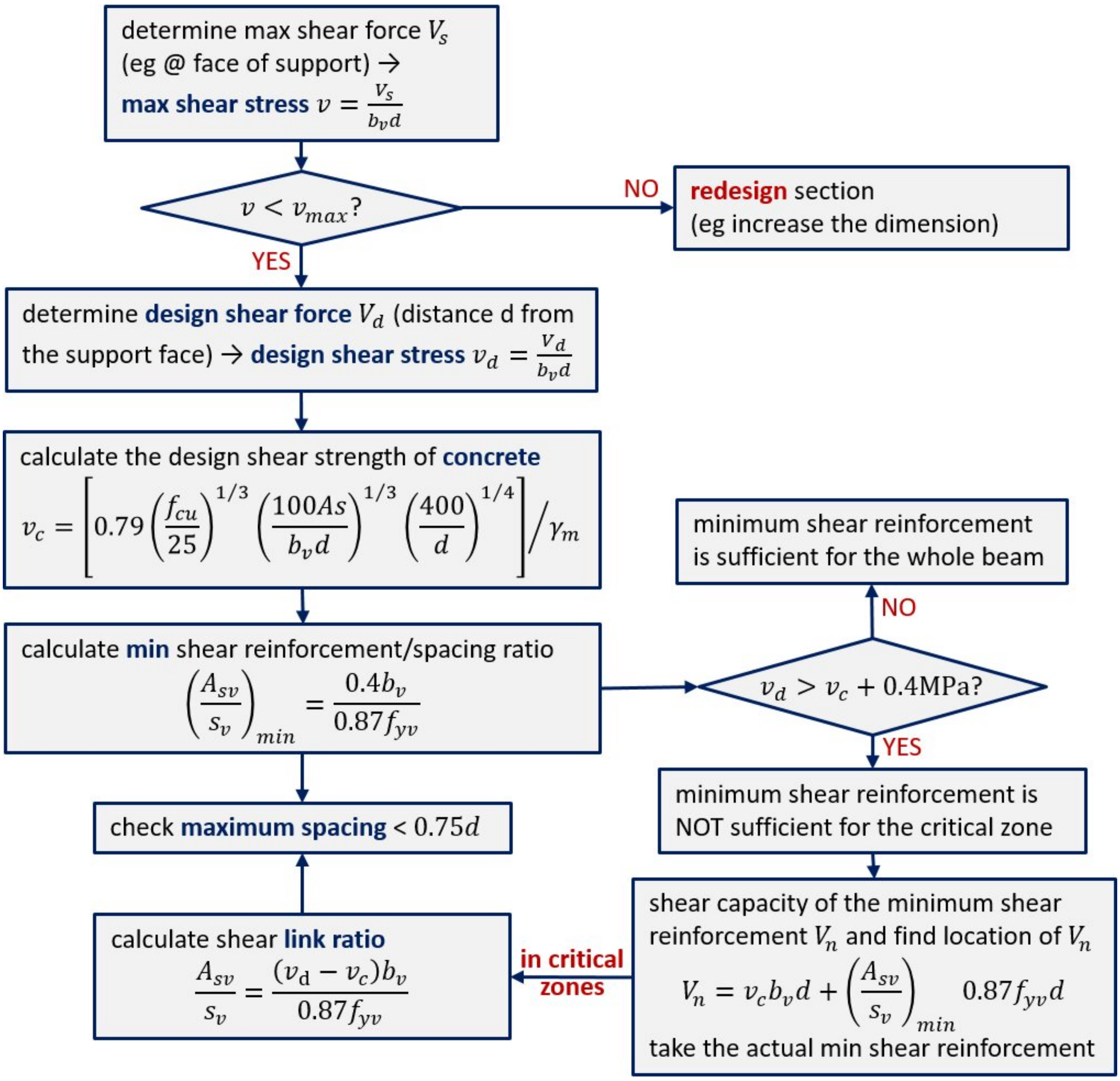

Figure 3.16: Shear design flowchart

3.5.2 Shear Design Flowchart

Shear design steps

1. Determine max shear force V_s and stress $v = \frac{V_d}{bd}$.
   The maximum shear force usually occurs at the face of the support.
2. Check Maximum Shear Stress $v < v_{max} = \min(0.8\sqrt{f_{cu}}, 7)$.
   If not, redesign section by increasing its dimensions.
3. Calculate the design concrete shear stress v_c.
   Check if at the nominal section $v_c + 0.4 < v$.
4. If yes, provide additional links to cover $v - v_c$ as $\frac{A_{sv}}{s_v} = \frac{(v - v_c)b}{0.87 f_{yv}}$.
5. If no, use minimum $\frac{A_{sv}}{s_v} = \frac{0.4b}{0.87 f_{yv}}$.
6. Ensure $s_v < 0.75d$.

3.5.3 Shear Resistance of Bent-Up Bars

In regions of high shear stresses, the use of links to carry the full shear force may cause steel congestion. In these situations, 'bending up' main longitudinal reinforcement that is no longer required to resist bending moments in the beam can be considered. Main reinforcing bars in beams may be bent up near the supports. Similar to the truss analysis, the bent-up bars and the concrete in compression are considered to act as an analogous lattice girder. Two cases can be distinguished (see Figure 3.17):

- Single system: when one diagonal crack intercepts one bent-up bar
- Multiple system: when one diagonal crack intercepts more than one bent-up bar

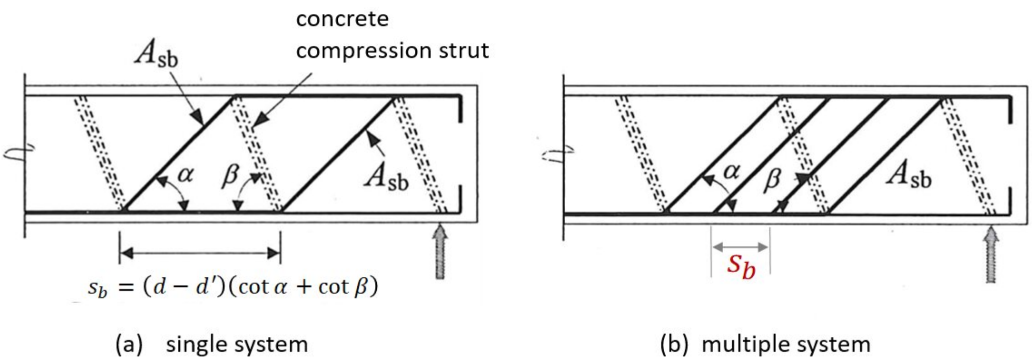

Figure 3.17: Single and Multiple system of bent-up bars.

Single System In a single system one diagonal crack intercepts only one bent-up bar. Let, the spacing of the bent-up bars be s_b, it holds (see Figure 3.17) that

$$s_b = (d - d')(\cot\alpha + \cot\beta)$$

where:

- α : angle between bent-up bar and axis of the beam
- β : angle between compression strut and axis of the beam

Accordingly, the shear resistance of a single bent-up bar is:

$$V_b = (0.87 f_{yv} A_{sb}) \sin\alpha \tag{3.12}$$

where

- A_{sb} : is the cross-section area of the bent-up bar

Multiple System In a multiple system one diagonal crack intercepts more than one bent-up bar. The shear resistance is given by:

$$V_b = 0.87 f_{yv} A_{sb} \sin\alpha \left(\frac{d-d'}{s_b}\right)(\cot\alpha + \cot\beta) \tag{3.13}$$

Where the **number of bent-up bars** that intercept the diagonal crack is

$$\left(\frac{d-d'}{s_b}\right)(\cot\alpha + \cot\beta)$$

The truss should be arranged such that:

- both α and β are $\geq 45°$
- spacing $s_b \leq 1.5d$

For instance, for

$$\alpha = \beta = 45° \quad \text{and} \quad s_b = (d-d'), V_b$$

we have what is known as a **'double system'** arrangement, since the number of bent-up bars crossing the diagonal crack is two:

$$\frac{(d-d')(\cot\alpha + \cot\beta)}{s_b} = \frac{(d-d')(\cot 45° + \cot 45°)}{d-d'} = 2$$

The resistance from the bent-up bars is then (Equation 3.5.3):

$$V_b = 0.87 f_{yv} A_{sb} \sin 45° \cdot 2 = 1.223 f_{yv} A_{sb}$$

- HKCC2013: clause 9.2.2 [7] states that at least 50% of the necessary shear reinforcement should be in the form of links.

3.5.4 Design Example: Bent-Up Bars

Determine the shear resistance of the beam of Figure 3.18 for shear reinforcement of both stirrups and bent-up bars. Assume:

- Material strengths: $f_{yv} = 250\,\mathrm{N/mm^2}$ for stirrups, $f_y = 500\,\mathrm{N/mm^2}$ for bent-up bars, respectively, and $f_{cu} = 30\,\mathrm{N/mm^2}$ for concrete.
- cross section geometry: $b = 350\,\mathrm{mm}$ and $d = 650\,\mathrm{mm}$
- Links: R12@$100\,\mathrm{mm}$, bent-up bars: 2T25 ($A_{sb} = 491\,\mathrm{mm^2}$) double system.

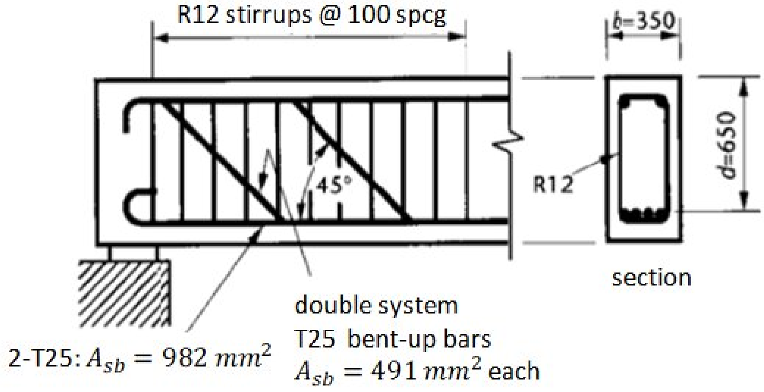

Figure 3.18: Beam with Stirrups and Bent-Up Bars

Solution

First we calculate the resistance of the cross-section with the stirrups. We start from the design concrete shear stress. For longitudinal reinforcement of 2T25:

$$\frac{100A_s}{bd} = \frac{100 \cdot 982}{350 \cdot 650} = 0.43$$

The design concrete shear stress is:

$$v_c = 0.79\left(\frac{f_{cu}}{25}\right)^{\frac{1}{3}}\left(\frac{100A_s}{b_w d}\right)^{\frac{1}{3}}\left(\frac{400}{d}\right)^{\frac{1}{4}}\frac{1}{\gamma_m} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{\frac{1}{3}}(0.43)^{\frac{1}{3}}(1)^{\frac{1}{4}} \rightarrow$$

$$v_c = 0.50\,\mathrm{N/mm^2}$$

For the R12@100 mm stirrups:

$$\frac{A_{sv}}{s_v} = \frac{226}{100} = 2.26\,\mathrm{mm^2/mm}$$

Thus, the shear resistance of the concrete V_c plus the stirrups V_s is:

$$V_{c+s} = V_c + V_s = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right)0.87 f_{yv} d$$

$$= (0.5 \cdot 350 \cdot 650 + 2.26 \cdot 0.87 \cdot 250 \cdot 550) \cdot 10^{-3} = 114 + 320 = 434\,\mathrm{kN}$$

The bent-up bars 2T25 $A_{sb} = 491\,\mathrm{mm^2}$ are arranged as a double system $\rightarrow$, for this arrangement the shear resistance of bent-up bars is:

$$V_b = 1.232 f_{yv} A_{sb} = 1.23 \cdot 500 \cdot 491 \cdot 10^{-3} = 302.0\,\mathrm{kN}$$

The shear resistance of the beam (the total shear resistance of the concrete + stirrups + bent-up bars) is:

$$V = V_{c+s} + V_b = 434 + 302 = 736\,\mathrm{kN}$$

Check: According to the code requirement $\geq 50\,\%$ of the necessary shear reinforcement should be in the form of links. Indeed, the shear resistance of links is $320\,\mathrm{kN} > 302\,\mathrm{kN}$ = the shear resistance of bent-up bars, satisfying HKCC2013.

Chapter 4

Torsion

Overview: This section covers the following topics:

- Introduction to torsional loading in R/C members, including sources like curved bridges and spandrel beams.
- Stress development and spiral crack formation due to torsional shear stresses in concrete.
- Design for torsional reinforcement using closed links and longitudinal bars, based on the space truss analogy [HKCC2013: clause 6.3.6].
- Calculation of torsional shear stress using elastic (soap-film) and plastic (Nadai's sand-heap) analogies.
- Design procedure for torsion in rectangular sections, including shear stress checks and reinforcement detailing requirements.
- Example of designing torsional reinforcement for a section with combined shear and torsion demands.

4.1 Intro

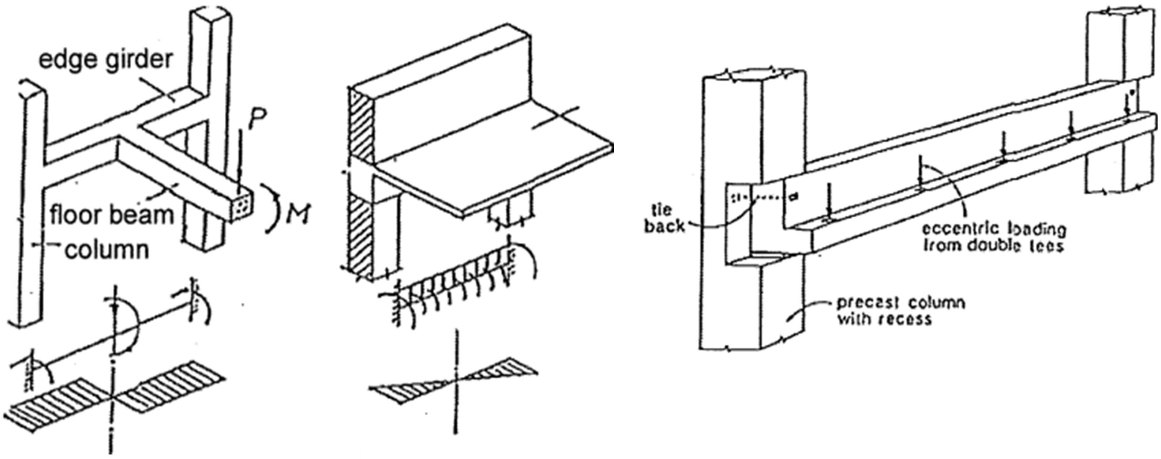

Figure 4.1: R/C members subjected to torsion e.g., edge beams

Typical cases of torsional loading
Reinforced Concrete (R/C) members are primarily designed to resist bending moments, transverse shear forces associated with these moments, and, in the case of columns, axial forces often combined with bending and shear (Figure 4.1). However, in certain structural configurations, torsional moments can also act, causing the member to twist about its longitudinal axis. Torsional loading typically arises in scenarios such as curved or skewed bridge girders, where the geometry induces twisting due to uneven load distribution; spandrel beams in building frames, which experience torsion from eccentric loading of adjacent slabs (Figure 4.2).

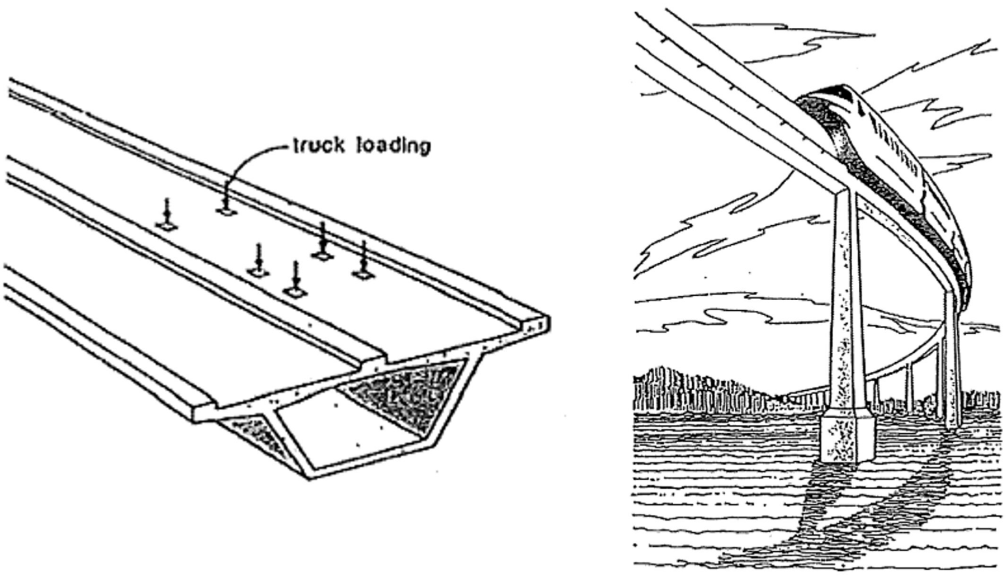

Figure 4.2: R/C members subjected to torsion, e.g., eccentrically loaded box-girder bridge

4.1.1 Stress Development and Crack Formation

Torsional moments, denoted as T, act to twist the beam about its longitudinal axis, inducing shear stresses within the concrete cross-section. These shear stresses are distributed such that they generate principal tensile stresses, which are inclined at approximately 45 degrees to the longitudinal axis. When the principal tensile stresses exceed the tensile strength of the concrete, diagonal cracks begin to form along the 45-degree planes (Figure 4.3). This cracking is a direct result of concrete's low tensile capacity, typically around 10% of its compressive strength, making it prone to failure under tension. As these diagonal cracks propagate and interconnect, they form a spiral pattern around the beam. This spiral cracking manifests as a three-dimensional (3D) cracking configuration, which is characteristic of torsional failure in plain concrete beams.

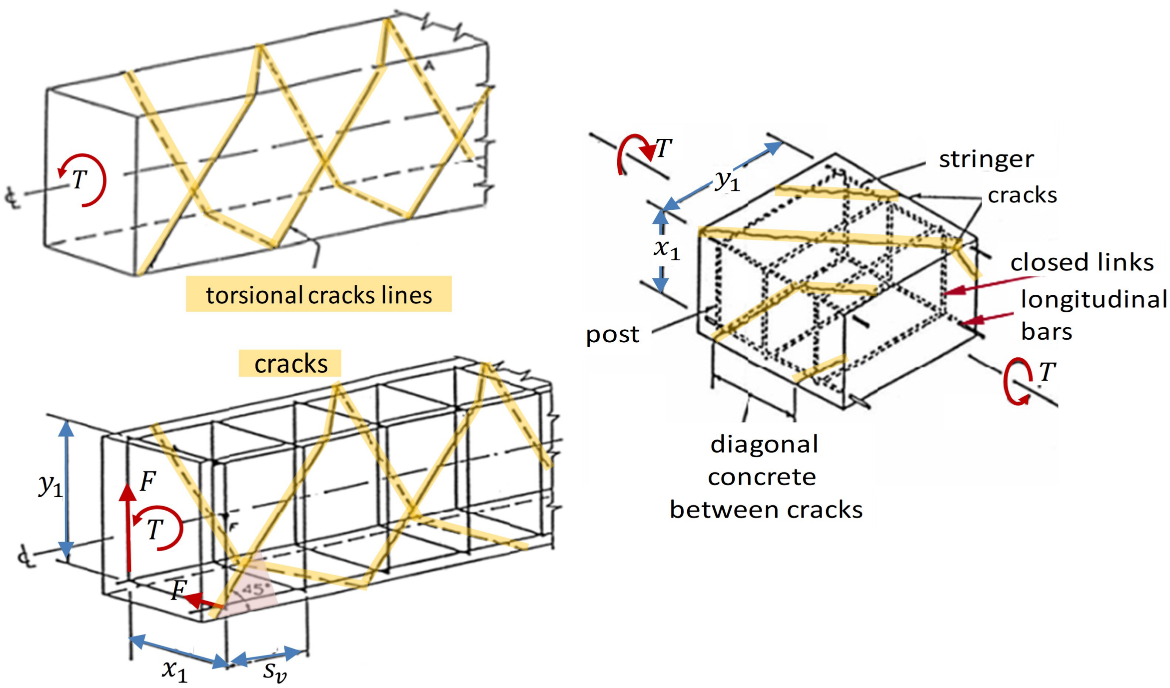

Figure 4.3: Diagonal cracking of plain concrete beam due to torsion. Space truss model used to design for torsion.

4.1.2 Design for Torsional Reinforcement

[HKCC2013: clause 6.3.6] When the torsional shear stress v_t exceeds the torsional capacity of the concrete section, torsional reinforcement must be provided to ensure structural integrity. This reinforcement is typically arranged as (1) **closed links** and (2) **longitudinal bars**, which work together with the concrete to resist increasing torsional moments after cracking through a space truss action. In this **space truss analogy**, the concrete acts as compressive struts between the links, while the steel reinforcement serves as tension members. Failure should ultimately occur by yielding of the reinforcement before the crushing of the concrete. For design purposes, we assume that the tension reinforcement in the form of closed links must be provided to resist the full torsional moment, ensuring the section's capacity to handle the applied torsion.

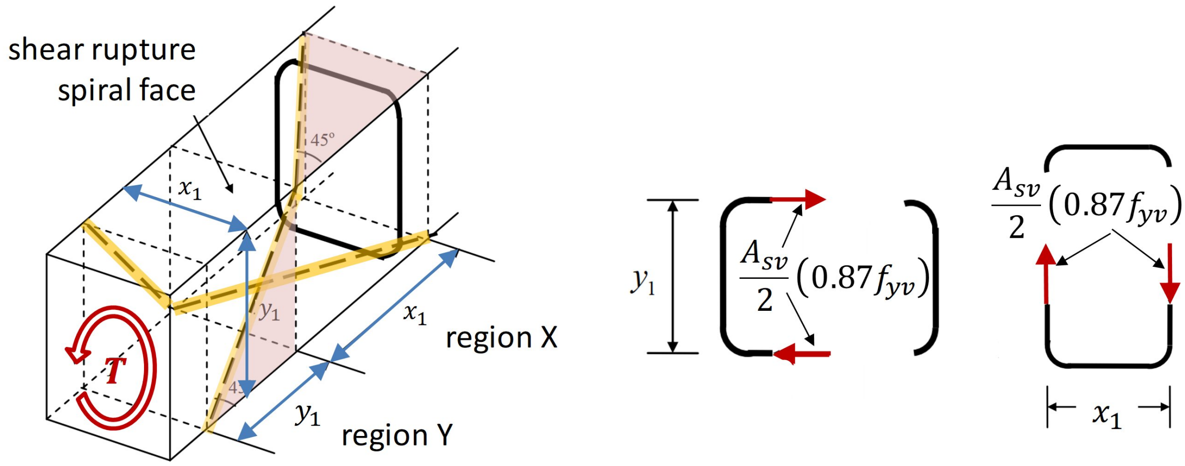

Figure 4.4: Space truss model used to design for torsion.

Design formulae

With reference to Figure 4.4, the basic **assumption** is that the shear rupture length (Kong & Evans, 1987 []) equals the stirrup width plus stirrup length $= x_1 + y_1$ Accordingly, the torsional (moment) resistance of closed links (stirrups) within regions X and Y is identical:

- Moment of one stirrup in region X:

$$\text{(force from vertical leg)} \cdot \text{(lever arm } = y_1) = \frac{A_{sv}}{2}(0.87f_{yv}) \cdot y_1$$

- number of stirrups within region X:

$$\frac{x_1}{s_v}$$

- Moment from all stirrups within region X:

$$\frac{A_{sv}}{2}(0.87f_{yv}) \cdot y_1 \cdot \frac{x_1}{s_v}$$

- Same for the moment of a stirrup in region Y:

$$\text{(force from horizontal leg)} \cdot \text{(lever arm } = x_1) = \frac{A_{sv}}{2}(0.87f_{yv}) \cdot x_1$$

- Total moment of resistance within regions X and Y:

$$A_{sv}(0.87f_{yv}) \cdot \frac{x_1 y_1}{s_v}$$

4.1.3 Design for Torsional Reinforcement

Note: Torsional reinforcement is provided [HKCC2013: clause 6.3.6 [7]] additional to design requirements for bending and shear.

1. **Closed links**
2. The torsional resistance of the closed links is:

$$T = 0.8 \cdot \frac{A_{sv}}{s_v} x_1 y_1 (0.87f_{yv}) \tag{4.1}$$

where:

- x_1: smaller dimension of the link
- y_1: larger dimension of the link
- A_{sv}: the area of a two-leg link
- s_v: link spacing

3. 0.8 = an efficiency factor to allow for errors in assumptions made about the space truss behavior

4. The torsional two-leg closed links must then be provided such that:

$$\frac{A_{sv}}{s_v} \geq \frac{T}{0.8 x_1 y_1 (0.87 f_{yv})} \tag{4.2}$$

Torsional longitudinal bars [HKCC2013: clause 6.3.6 [7]]

- Torsional longitudinal reinforcement resists the longitudinal component of the diagonal tension forces of the space truss.
- Consider a length s_v of the beam: the torsional longitudinal bars should be such that the total quantity = same volume as the steel in the links:

$$A_{sv} = \frac{A_{sv}}{2}(x_1 + y_1)$$

- It is desirable that the torsional longitudinal bars and the closed links should yield simultaneously. To achieve this we set the steel volume-strength products of the longitudinal bars and the links equal:

$$A_{sv} \cdot f_y = A_{sv}(x_1 + y_1) f_{yv} \rightarrow A_{sv} f_y = \frac{A_{sv} f_{yv}(x_1 + y_1)}{s_v} \rightarrow$$

- The torsional longitudinal bars required is

$$A_s \geq \frac{A_{sv}}{s_v} \frac{f_{yv}}{f_y}(x_1 + y_1)$$

4.1.4 Torsional Shear Stress Calculation

An Elastic Method of Analysis: Soap-film Analogy

The membrane analogy, which applies to elastic bodies, can be used to visualize the distribution of shearing stresses on a cross section. The equations for the slope of an inflated membrane are analogous to those for shearing stress due to torsion. Thus, the distribution of shearing stresses can be visualized by cutting an opening in a plate proportional to the shape of the cross section loaded in torsion, stretching a membrane or soap film over this opening, and inflating the membrane.

A Plastic Method of Analysis: Nadai's Sand-Heap Analogy

In fully plastic bodies, the shearing stress remains uniform across all points. Consequently, the soap-film analogy, which assumes variable slopes, must be replaced by a figure with a constant slope corresponding to the fully plastic state. This results in a cone for a circular shaft or a pyramid for a square member. Such shapes can be visualized by pouring sand onto a plate matching the cross-section's geometry. However, the torsional behavior of uncracked concrete members is neither perfectly elastic, as modeled by the soap-film analogy, nor perfectly plastic, as represented by the sand-heap analogy. Nevertheless, solutions derived from both models have been effectively applied to predict torsional behavior.

- A plastic analysis is recommended for calculating the torsional shear stress v_t, where the expressions are derived from Nadai's sand-heap analogy in the theory of plasticity.
- Torsional shear stress $v_t = \tan\theta$ thus

$$v_t = \frac{2\alpha}{h_{\text{min}}}$$

- In the sand heap for a rectangular section, h_{min} and h_{max} are the smaller and larger dimensions of the section, respectively.

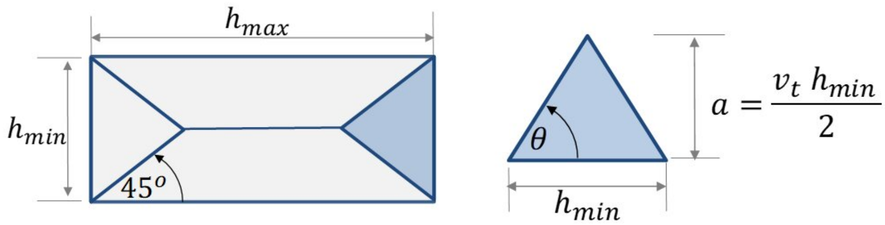

Figure 4.5: Nadai's sand-heap for a rectangle section: plane view (left), end view (right)

- Ultimate torsional moment:

$$T = 2\,[\text{“the volume of the sand heap”}]$$
$$\frac{T}{2} = 2 \cdot \frac{1}{3} h_{\text{min}} \cdot \frac{1}{2} h_{\text{min}} a + \frac{1}{2} h_{\text{min}} a (h_{\text{max}} - h_{\text{min}})$$
$$\frac{T}{2} = \frac{1}{4} v_t\, h_{\text{min}}^2 \left(h_{\text{max}} - \frac{1}{3} h_{\text{min}} \right)$$

- Torsional reinforcement must be provided when the torsional shear stress v_t exceeds the torsional capacity of the concrete section.
- Torsional capacity of concrete sections limited to $0.60\,\mathrm{N/mm^2}$ [HKCC2013: clause 6.3.4 [7]]

$$v_{\text{cmin}} = \min\left(0.067\sqrt{f_{cu}}, 0.60\,\mathrm{N/mm^2}\right)$$

4.1.5 Design procedure for torsion of a rectangular section

Reinforcement Detailing Requirements [HKCC2013: clauses 6.3.7 and 6.3.8 [7]] In general, the reinforcement in a beam should include the reinforcement requirement for flexure plus shear plus torsion. The design procedure for torsion is:

1. Calculate the torsional shear stress using:

$$v_t = \frac{2T}{h_{\text{min}}^2 \left(h_{\text{max}} - \frac{h_{\text{min}}}{3} \right)}$$

where:

- T: torsional moment due to ultimate load;
- h_{max} and h_{min}: the larger and smaller dimensions of the section, respectively.

2. Check the torsional shear stress v_t:

   i. when $v_t > v_{t\text{min}} = \min\left\{0.067\sqrt{f_{cu}}, 0.6\,\mathrm{N/mm^2}\right\}$, torsional reinforcement must be provided

   ii. when $y_1 \le 550\,\mathrm{mm}$, the section is considered as a small section and

   $$v_t \le v_{tu}\frac{y_1}{550}$$

   This condition is to prevent spalling of the corners of sections.

   iii. Torsion is seldom present alone, and will normally be combined with direct shear stress in most practical situations. The sum of the shear stresses from direct shear stress $v = \frac{V}{b_v d}$ and torsional shear stress v_t must NOT exceed the ultimate torsional shear stress [HKCC2013: clause 6.3.4 [7]]:

   $$(v + v_t) \le v_{tu} = \min\left\{0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}\right\}$$

   when the combined shear stress $(v + v_t) > v_{tu}$, the design is inadmissible and the cross-section dimensions should be increased

3. Reinforcement links for direct shear v and torsional shear v_t may be required according to Table 3.4.4.

4. Calculate the required torsional reinforcement in the form of closed links:

   $$\frac{A_{sv}}{s_v} \ge \frac{T}{0.8x_1y_1(0.87f_{yv})}$$

   The requirements of reinforcement detailing are given in "Design for Torsional Reinforcement" [HKCC2013: clause 6.3.6 [7]]. The particular attention should be given to the link spacing:

   $$s_v \le \min\left\{200\,\mathrm{mm}, x_1, \frac{y_1}{2}\right\}$$

5. Calculate the required torsional longitudinal reinforcement:

   $$A_s \ge \frac{A_{sv}f_{yv}(x_1 + y_1)}{s_v f_y}$$

   - additional A_s (NOT the longitudinal reinforcement for bending) should be **distributed evenly around the inside perimeter of the links**
   - The clear distance between torsional longitudinal bars $\le 300\,\mathrm{mm}$ and ≥ 4 bars (one in each corner) should be used.

6. A flanged section should be divided into component rectangles and each component designed to carry a torsional moment given by:

   $$T_t = T\left[\frac{(h_{\text{min}}^2 h_{\text{max}})}{\sum(h_{\text{min}}^2 h_{\text{max}})}\right]$$

Recall that v_r is the shear stress from minimum links:

$$v_r = 0.4\,\mathrm{N/mm^2} \quad \text{for } f_{cu} \le 40\,\mathrm{N/mm^2}$$

or

$$v_r = 0.4\left(\frac{f_{cu}}{40}\right)^{\frac{2}{3}} \quad \text{for } 40 < f_{cu} \le 80\,\mathrm{N/mm^2}$$

	$v_t \leq v_{tmin}$	$v_t > v_{tmin}$
$v \leq v_c + v_r$	minimum shear reinforcement; NO torsion reinforcement	designed torsion reinforcement, but not less than the minimum shear reinforcement
$v > v_c + v_r$	designed shear reinforcement; NO torsion reinforcement	designed shear and torsion reinforcement

Reinforcement for direct shear v and torsional shear v_t. Note: the value of v_r is defined in Table 3.2-2, NOTES 2

4.1.6 Example: Design for Torsional Reinforcement

Design the cross-section of Figure 4.6 for torsion, assuming:

- A design shear force = $160\,\mathrm{kN}$, and a torsional moment = $10\,\mathrm{kN\,m}$
- Assume that the demand for tension steel as calculated from the design for bending (not shown) is $A_s = 1000\,\mathrm{mm}^2$
- Material strengths $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$, $f_{yv} = 250\,\mathrm{N/mm^2}$

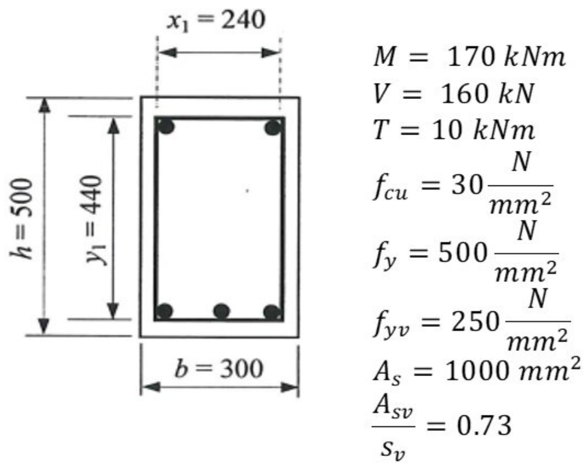

Figure 4.6: Example: design for torsional reinforcement

Solution

The design for torsion follows the design for bending and shear. We are given the outcome of the design for bending, therefore we start with shear design.

Shear strength of concrete:

The design shear concrete strength is:

$$A_s = 1000\,\mathrm{mm}^2 \quad \rightarrow \quad \frac{100A_s}{b_v d} = \frac{100 \cdot 1000}{300 \cdot 470} = 0.71$$

$$v_c = 0.79 \left(\frac{f_{cu}}{25}\right)^{1/3} \left(\frac{100A_s}{b_v d}\right)^{1/3} \left(\frac{400}{d}\right)^{1/4} = \frac{0.79}{1.25} \cdot 1.2^{\frac{1}{3}} \cdot 0.71^{\frac{1}{3}} \cdot 1^{\frac{1}{4}} = 0.60\,\mathrm{N/mm^2}$$

Direct shear stress demand:

$$v = \frac{V}{b_v d} = \frac{160 \cdot 10^3}{300 \cdot 470} = 1.13\,\mathrm{N/mm^2}$$

$$v = 1.13\,\mathrm{N/mm^2} > v_c + v_r = 0.60\,\mathrm{N/mm^2} + 0.4\,\mathrm{N/mm^2} = 1.00\,\mathrm{N/mm^2}$$

Shear link demand from direct shear:

$$\left(\frac{A_{sv}}{s_v}\right)_v = \frac{(v - v_c)b_v}{0.87 f_{yv}} = \frac{(1.13 - 0.60) \cdot 300}{0.87 \cdot 250} = 0.74$$

The ultimate torsional shear stress:

$$y_1 = 440\,\mathrm{mm} < 550\,\mathrm{mm} \rightarrow \text{the section is a small section}$$

Check: $v_{tu} = 0.8\sqrt{f_{cu}} = 0.8 \cdot \sqrt{30} = 4.38\,\mathrm{N/mm^2}$

$$v_t = 0.56\,\mathrm{N/mm^2} \leq v_{tu}\frac{y_1}{550} = 4.38\,\mathrm{N/mm^2} \cdot \frac{440}{550} = 3.5\,\mathrm{N/mm^2}$$

as required

The **total shear stress demand** comes from the sum of the direct shear stress $v = 1.13\,\mathrm{N/mm^2}$ and the torsional shear stress $v_t = 0.56\,\mathrm{N/mm^2}$:

$$(v + v_t) = 1.13\,\mathrm{N/mm^2} + 0.56\,\mathrm{N/mm^2} = 1.70\,\mathrm{N/mm^2} < v_{tu} = 4.38\,\mathrm{N/mm^2}$$

hence the design is admissible.

$$v = 1.13\,\mathrm{N/mm^2} > v_c + v_r = 0.60\,\mathrm{N/mm^2} + 0.4\,\mathrm{N/mm^2} = 1.00\,\mathrm{N/mm^2}$$

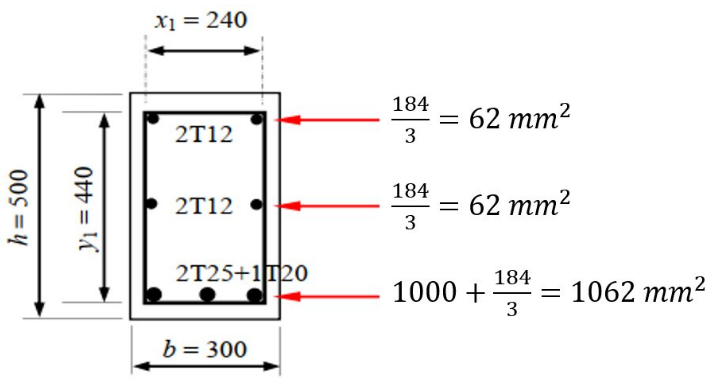

Figure 4.7: Torsional reinforcement arrangement

Links for torsional shear stress:

$$\frac{A_{sv}}{s_v} = \frac{T}{0.8 x_1 y_1 (0.87 f_{yv})} = \frac{10.0 \cdot 10^6}{0.8 \cdot 240 \cdot 440 \cdot 0.87 \cdot 250} = 0.54\,\mathrm{mm}$$

Total shear link demand link

$$\frac{A_{sv}}{s_v} = \left(\frac{A_{sv}}{s_v}\right)_v + \left(\frac{A_{sv}}{s_v}\right)_t = 0.73 + 0.54 = 1.27\,\mathrm{mm}$$

Provide R10@100 (ie R10 closed sher links at 100 mm spacing) with 2 legs of the closed type with their ends fully anchored: $A_{sv}/s_v = 1.57$

Torsional longitudinal steel

$$A_s = \frac{A_{sv} f_{yv}}{s_v f_y}(x_1 + y_1) = 0.54\frac{250}{500}(240 + 440) \rightarrow A_s = 184\,\mathrm{mm}^2$$

Total main longitudinal reinforcement:

$$A_s = 1000(\text{flexure}) + \frac{184}{3}(\text{torsion long reinf.}) = 1062\,\mathrm{mm}^2$$

Provide 2T25+1T20 $\left(A_s = 1295\,\mathrm{mm}^2\right)$

4.2 FAQs on Shear and Torsion Design

This is a collection of frequently asked questions (FAQs) related to the design of reinforced concrete sections under shear and torsion, compiled based on common student inquiries and clarifications provided during lectures.

Q1: Which shear force should be used for the design of a multi-span beam?

Shear reinforcement should be provided so to cover the shear force demand (as prescribed by the shear forced envelope) in all sections of the structure. Usually, the design flow of beams starts from the critical section (located at distance distance d from the support face) and reducing the shear reinforcement in the midspan of the beam.

Q2: What happens if the design shear stress v is less than $v_c + 0.4$?

If the concrete capacity plus the minimum reinforcement capacity (0.4 N/mm^2) exceeds the design shear stress, only **minimum shear reinforcement** is required throughout that region.

Q3: How do we determine the region where shear reinforcement can be reduced to the minimum?

Practically, you identify the point along the beam where the design shear force drops below the resistance provided by the minimum shear reinforcement (typically $v_c + 0.4$). When calculating this transition point, use the actual provided bar diameter and spacing rather than theoretical required values to determine the exact shear force capacity. In all regions where the demand is lower than this threshold, minimum stirrups are sufficient.

Q4: Which tension steel area A_s should be used to calculate v_c?

Use the area of tension reinforcement at the section currently being checked for shear. For support regions, this is typically the reinforcement at a distance d from the face of the support. In cross sections where it is unclear which is the tension steel (top or bottom), be conservative and use the area of smaller reinforcement ares at the section.

Q5: How is the longitudinal steel for torsion calculated?

Additional longitudinal reinforcement for torsion is calculated based on the torsional shear reinforcement-to-spacing ratio ($A_{sv,t}/s_v$). Refer to Lecture 10, slides 24–25 for the specific equations.

Q6: What if the section does not require additional torsion reinforcement?

If the torsional stress is below the threshold requiring reinforcement, no additional links or longitudinal bars are needed specifically for torsion. You only provide reinforcement required for flexure and shear.

Q7: What does f_{yv} represent?

f_{yv} is the characteristic yield strength of the steel used specifically for shear reinforcement (i.e., the stirrups or links). Designers commonly choose between 250 N/mm^2 and 500 N/mm^2. If the value is not specified in a problem, 250 N/mm^2 is typically adopted as a common design practice.

Q8: What are v_c and V_r in shear design?

v_c (or V_c) is the design concrete shear stress, representing the capacity of the concrete itself to resist shear. v_r (or V_r) is the shear resistance provided by the steel links or stirrups.

Q9: Are there any drawbacks to placing too much shear reinforcement?

Yes. Beyond economic inefficiency, high reinforcement density increases construction complexity. It becomes difficult to maintain the required clear distance between bars, making it hard to properly place and consolidate the concrete. This can lead to bond issues and voids.

Q10: How do I calculate A_{sv} for a stirrup with multiple legs?

$A_{sv} = n \cdot \frac{\pi\phi^2}{4}$, where n is the number of legs. For example, a 2-legged $R10$ stirrup ($\phi = 10$mm) has $A_{sv} = 2 \cdot \frac{3.14 \cdot 10^2}{4} \approx 157$ mm^2.

Chapter 5

Anchorage, Bond & Curtailment

Overview: This chapter covers the following topics:

- Fundamentals of bond and anchorage, including the mechanical interaction between steel reinforcement and concrete.
- Calculation of anchorage length using bond stress ($f_{bu} = \beta\sqrt{f_{cu}}$) and ultimate anchorage length coefficient (K_A).
- Anchorage methods: straight bars, hooks, and bends, with effective anchorage length (l_e) for hooks and bends.
- Lapping of reinforcement, including rules for tension and compression lap lengths and adjustments for location-specific conditions.
- Use of mechanical couplers as an alternative to lapping for bar continuity.
- Crack control through bar spacing limits and side face reinforcement in large beams.
- Curtailment of reinforcement bars to optimize steel use, with methods including capacity checking and standard rules for simply supported and continuous beams.

5.1 Anchorage and Bond Between Steel and Concrete

In general, anchorage and bonding between steel and concrete is a complex mechanical issue which however, is simplified by design codes so its treatment becomes simple and practical [3, 16]. So far, we have assumed steel force transfers to the cross-section without concern. We now address how this transfer occurs.

Bond is the grip due to *adhesion* or *mechanical interlock* and bending in deformed bars between the reinforcement and the concrete. **Anchorage** is the embedment of a bar in concrete so that it can carry load through bond between the steel and the concrete. The reinforcing bar subject to direct tension must be firmly anchored so it is not to be pulled out of the concrete. If the anchorage length is sufficient, the full strength of the bar can be developed by bond.

The anchorage depends mainly on

1. the bond between the bar and the concrete,
2. the area of contact.

The force in the bar should be transfered to the surrounding concrete along an appropriate embedment length l, i.e., the **anchorage bond length** in the concrete (often simply called the **anchorage length**). The force in the bar is given by:

$$F_s = \frac{\pi \phi_e^2}{4} f_s$$

where:

- f_s: is the direct axial (tensile or compressive) stress in the bar,
- ϕ_e: is the bar size (diameter of the bar).

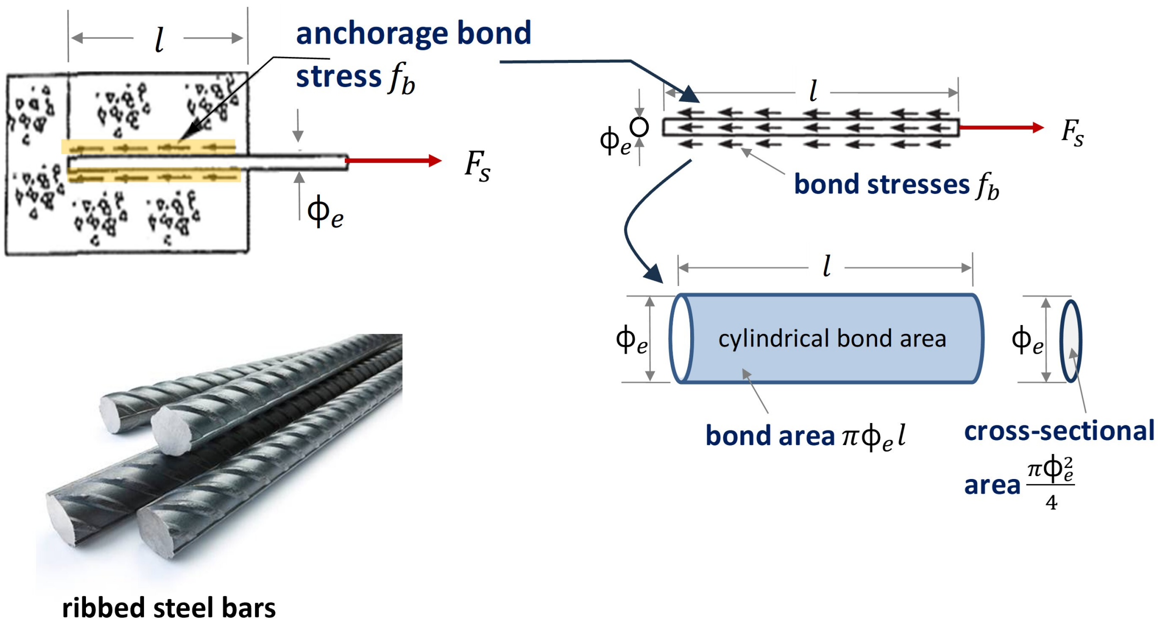

Figure 5.1: Anchorage bond length in the concrete, often called 'anchorage length'

Figure 5.1 visualizes the bond stresses between the bar and the surrounding concrete. The basic assumption is that the anchorage bond stress (simply called **anchorage stress**) is assumed to be uniform and equal to f_b over the anchorage length l. This is a simplifying assumption as the bond stress distribution is nonuniform in reality. Accordingly, the anchorage force is $F_b =$ (cylindrical area times bond stress) $= (\pi\phi_e l)f_b$ where f_b is the design anchorage bond stress ([HKConcrete Eq. 8.2]). The anchorage length is found by equating the force in the bar with the bond force $F_s = F_b$:

$$\frac{\pi\phi_e^2}{4}f_s = (\pi\phi_e l)f_b \rightarrow l = \frac{f_s}{4f_b}\phi_e$$

The **ultimate anchorage length** l_b (often referred to as the full anchorage length) is the length of the reinforcing bar required to develop the full strength of the bar, which corresponds to the maximum stress in the bar $f_s = 0.87f_y$. There are two (practically equivalent) methods to calculate it:

1. $f_{bu} = \beta\sqrt{f_{cu}}$ is the design ultimate anchorage bond stress (Table 5.1), or
2. as multiples of bar diameter with the aid of the k_A coefficient (Table 5.2).

With the aid of the design ultimate anchorage bond stress $f_{bu} = \beta\sqrt{f_{cu}}$, the **ultimate anchorage length** l_b is:

$$l_b = \frac{0.87f_y}{4f_{bu}}\phi_e \tag{5.1}$$

where:

- when calculating f_{bu} the characteristic compressive cube strength of concrete, f_{cu} is limited to $60\,\mathrm{N/mm^2}$ [HKCC2013 [7]: 8.4.3]
- $\beta =$ a coefficient that depends on the bar type given from Table 5.1,
- ribbed bars are commonly used in practice, so assume by default ribbed bars unless explicitly stated otherwise.

bar type	β	
	in tension	**in compression**
plain bars	0.28	0.35
ribbed bars	0.50	0.63
fabric	0.65	0.81

Table 5.1: Values of **bond coefficient** β [HKCC2013: Table 8.3]

Alternatively, using K_A the **ultimate anchorage length** l_b can be estimated directly as multiples of bar diameter:

$$l_b = K_A\phi \tag{5.2}$$

where the ultimate anchorage length coefficient K_a is obtained from Table 5.2 based on concrete grade, steel grade, and bar surface type.

5.1.1 Example: design of anchorage required for straight bars

Calculate the anchorage length in tension, assuming:

- Bar diameter: $\phi = 25\,\mathrm{mm}$
- Beam depth: $h = 400\,\mathrm{mm}$
- Material strengths: $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$

Concrete Grade	Type of Anchorage Length	Reinforcement Types		
		$f_y = 250\,\mathrm{N/mm^2}$ (ribbed)	$f_y = 500\,\mathrm{N/mm^2}$ (ribbed)	Fabric
30	Tension	36	40	31
	Compression	29	32	25
35	Tension	33	38	29
	Compression	27	30	23
40	Tension	31	35	27
	Compression	25	28	22
45	Tension	29	33	25
	Compression	24	26	20
50	Tension	28	31	24
	Compression	22	25	19
≥ 60	Tension	26	28	22
	Compression	20	23	18

Table 5.2: **Ultimate anchorage length coefficient K_A** [HKCC2013: Table 8.4]

Solution #1 (ultimate anchorage length coefficient K_A method)

The ultimate anchorage bond length is $l_b = K_A \phi$:

- From Table 5.2, for ribbed bars, assuming tension which is the worst case, concrete grade 30, steel grade 500: $k_a = 40$.
- Anchorage length: $l = 40 \cdot 25 = 1000\,\mathrm{mm}$.
- take $d = h - 50 = 400 - 50 = 350\,\mathrm{mm}$ $l_b = K_A \phi = 40 \cdot 25 = 1000\,\mathrm{mm}$
- Adjust for support: the effective span of a cantilever is (clear span + $d/2$) $\rightarrow$ hence the total anchorage length required is

$$l_b + \frac{d}{2} = 1000 + \frac{350}{2} = 1175\,\mathrm{mm}$$

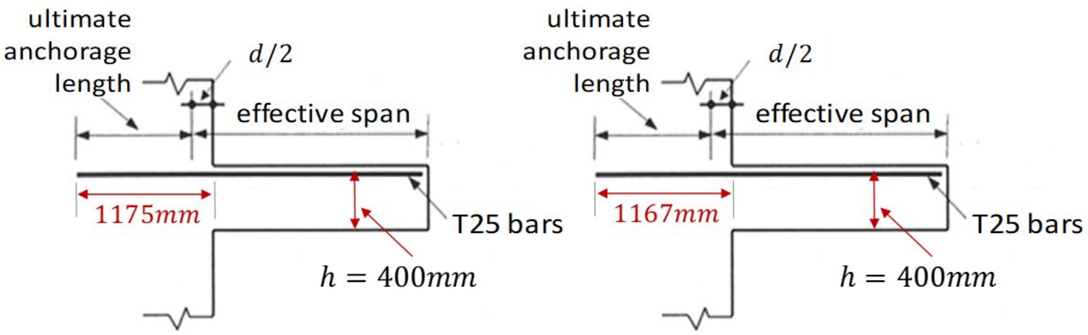

Figure 5.2: Ultimate anchorage length accodring to the coefficient K_A method (left) and β method(right).

solution #2 (bond coefficient β method)

The ultimate anchorage bond length is calculated with Equation 5.1:

- The bond coefficient for bars in tension: $\beta = 0.5$ (ribbed bar, tension)

- The ultimate anchorage stress:

$$f_{bu} = \beta\sqrt{f_{cu}} = 0.5\sqrt{30} = 2.74\,\mathrm{N/mm^2}$$

- The ultimate anchorage length:

$$l_b = \frac{0.87 f_y}{4 f_{bu}}\phi = \frac{0.87 \cdot 500}{4 \cdot 2.74} \cdot 25 = 992\,\mathrm{mm}$$

- Hence the total anchorage length l_b required is:

$$l_b + \frac{d}{2} = 992 + \frac{350}{2} = 1167\,\mathrm{mm}$$

- The difference between the two methods is negligible $1175\,\mathrm{mm}$ *vs* $1167\,\mathrm{mm}$.

5.1.2 Anchorage by Hooks or Bends

When the space for a straight length is insufficient (e.g., column < 1.2 m), we use bends or hooks to shorten the length required for anchorage. However, bends and hooks do not contribute to compression anchorage. The calculation of the required anchorage length in the case of bends and hooks hinges on the concept of effective anchorage length. The **Effective Anchorage Length** l_e of a hook or bend is the length of the straight bar which has the same anchorage value. The anchorage is measured from that portion of the bar between the start of the bend and a point 4 times the bar size 4ϕ beyond the end of the bend.

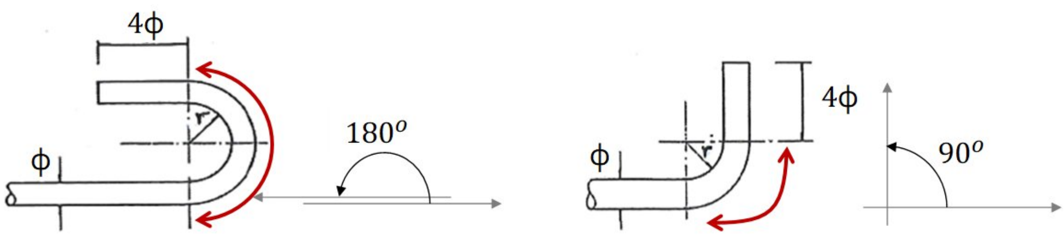

Figure 5.3: standard hook (180°) and bend (90°), r = internal radius.

Effective Anchorage Length l_e of a Hook or Bend

1. For a 180° hook:

$$l_e = 8r \le 24\phi \quad \text{(for high yield bars, usually take } l_e = 24\phi\text{)}$$

   or the actual length of the hook including the straight portion, whichever is greater.

2. For a 90° bend:

$$l_e = 8r \le 12\phi \quad \text{(for high yield bars, usually take } l_e = 12\phi\text{)}$$

   or the actual length of the bar, whichever is greater.

3. Internal radius r in (1) and (2):

   - For mild steel bars: $r \ge 2\phi$ (usually take $r = 2\phi$)

- For high yield bars:
  - $\phi \leq 20\,\mathrm{mm},\ r \geq 3\phi$ (usually take $r = 3\phi$)
  - $\phi > 20\,\mathrm{mm},\ r \geq 4\phi$ (usually take $r = 4\phi$)

4. Any length of the bar in excess of 4ϕ beyond the end of the bend may also be included for effective anchorage.

5.1.3 Example: design of anchorage required for hook bars

The ultimate anchorage length coefficient is determined using Table 5.2, which provides coefficients for different concrete grades and reinforcement types under tension and compression. For this solution, we focus on the worst-case scenario: **tension**, as it typically requires a longer anchorage length due to higher bond stress demands.

The calculated ultimate anchorage length for this case is $1175\,\mathrm{mm}$.

- Bar diameter: $\phi = 25\,\mathrm{mm}$
- Beam depth: $h = 400\,\mathrm{mm}$
- material strengths: $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$
- Calculate the anchorage length in tension.

Solution #1 (ultimate anchorage length coefficient K_A method)

The ultimate anchorage bond length is $l_b = K_A\phi$:

- From Table 5.2, for ribbed bars, assuming tension which is the worst case, concrete grade $f_{cu} = 30\,\mathrm{N/mm^2}$), steel grade $f_y = 500\,\mathrm{N/mm^2}$) $\rightarrow$ Ultimate anchorage length coefficient: $K_A = 40$.
- take $d = h - 50 = 400 - 50 = 350\,\mathrm{mm}$ $l_b = K_A\phi = 40 \cdot 25 = 1000\,\mathrm{mm}$
- The effective anchorage length of a standard 180° hook:

$$l_e = 24\phi = 24 \cdot 25$$

- The straight part of the bar is:

$$K_A\phi + \frac{d}{2} - l_e = 40 = (40 \cdot 25 + \frac{350}{2}) - 24 \cdot 25 = 575\,\mathrm{mm}$$

- Internal radius r for take:

$$r = 4\phi = 100\,\mathrm{mm} \quad \text{(high yield bars, } \phi > 20\,\mathrm{mm})$$

- Beyond the end of the bend:

$$4\phi = 100\,\mathrm{mm}$$

- Curved length (central line):

$$\pi(r + \frac{\phi}{2}) = 353\,\mathrm{mm}$$

- Total anchorage length required (see Figure 5.4) is:

$$575\,\mathrm{mm}\text{(straight length)} + 353\,\mathrm{mm}\text{(curved length)} + 100\,\mathrm{mm}(= 4\phi) = 1028\,\mathrm{mm}$$

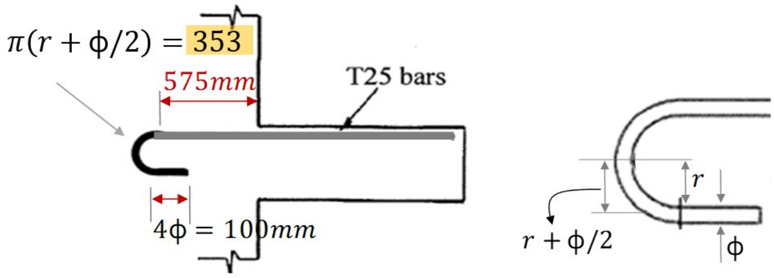

Figure 5.4: standard hook (180°) and bend (90°), r = internal radius.

5.1.4 Lapping of Reinforcement

- Issue: Bars are limited to $12\,\mathrm{m}$ (transportation constraint); spans (e.g., $15\,\mathrm{m}$) require multiple bars.
- Location: Prefer staggered laps away from high-stress sections (e.g., mid-span with low bending moment).
- Lap length: Generally equals tension anchorage length, with penalties:
  - Top of section: +40
  - Corners (2*d* from edge): +40
  - Top and corner: +200
- Compression lap: At least 1.25 × design compression anchorage length.

Laps in Reinforcement Determine the anchorage length required for the lengths of reinforcing bars that are joined by lapping to transfer the forces from one bar to another

The Rules for Lapping:

1. The laps should preferably be staggered and away from sections with high stresses.
2. The minimum lap length $>$ max $\{15\phi, 300\,\mathrm{mm}\}$ where ϕ = bar diameter [HKCC2013: 8.7.3.1].
3. Requirements for tension lap length: The lap length is not to be less than and generally equal to the design tension anchorage length [HKCC2013: Table 5.2].
   - At the top of a section and with minimum cover $< 2\phi$, the lap length is to be increased by 1.4
     Reason: The concrete at the top of a member is generally less compacted and also tends to have a greater water content, resulting in a lower concrete strength $\rightarrow$ direction of concreting effect.
   - At corners where minimum cover to either face $< 2\phi$ or clear spacing between adjacent laps $< 75\,\mathrm{mm}$ or 6ϕ, the lap length is to be increased by 1.4 *Reason:* At corners there is less confinement of the reinforcement
   - If conditions (b) and (c) both apply, the lap length is to be increased by 2.0.
4. The compression lap length should be at least 1.25 times the design compression anchorage length
5. All lap lengths are based on the smaller bar diameter

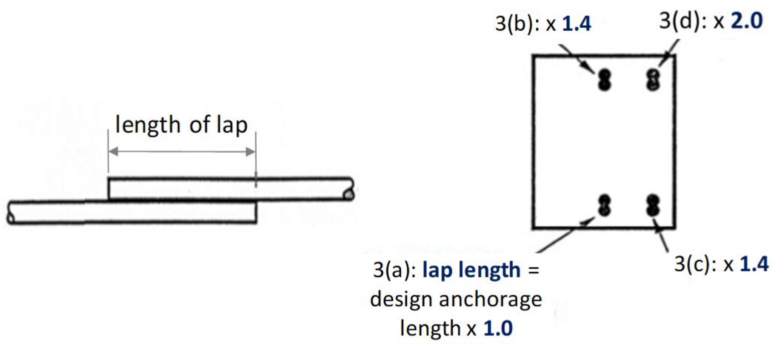

Figure 5.5: Left: lap of steel bars. Right: increased lap length depending on the location of the bar in the cross-section.

Mechanical Couplers

- Alternative to lapping: Bolt bars into a coupler for mechanical continuity.
- Pros: Avoids lapping; Cons: Higher cost (threading, bolting) and harder to inspect.
- Decision: Depends on project details—lapping is cheaper and easier to inspect; couplers suit specific cases.

Notes on Table 6.6:

- (1) for exposure condition 1, crack width has no influence on durability and this limit is set to guarantee acceptable appearance. In the absence of appearance conditions this limit may be relaxed
- water retaining structures referred to here are water tanks and the like used in general (2) building works and not meant to include large civil water retaining structures
- BS 8110: Part 1: Clause 3.12.11.2.1 and Part 2: Clauses 3.2.4 state that the maximum acceptable value of surface crack widths is $0.3\,\mathrm{mm}$ in normal environments

Crack control by limiting reinforcing bar spacing

In practical design, it is usual to comply with the $0.3\,\mathrm{mm}$ crack-width limit by a straightforward procedure of limiting the maximum distance between bars in tension.
Detailing rules for crack control are conveniently summarised with reference to Figure 5.6:

- Horizontal spacing between neighboring bars: $a_b \leq 150\,\mathrm{mm}$
- Corner distance $a_c \leq \frac{1}{2}a_b$
- in measuring the values of a_b and a_c, ignore any bars with a size < 0.45 times that of the largest bars (0.45, but not 0.5, is adopted so that, say, size 12 bars may be used with size 25 bars)
- for the maximum permissible centre-to-centre side bar spacing, $s_{b,max}$ of $250\,\mathrm{mm}$, minimum sizes of the side bars are $0.75\sqrt{b}$ for high-yield steel and $1.0\sqrt{b}$ for mild steel
- the minimum sizes are to guard against the bar yielding locally at a crack

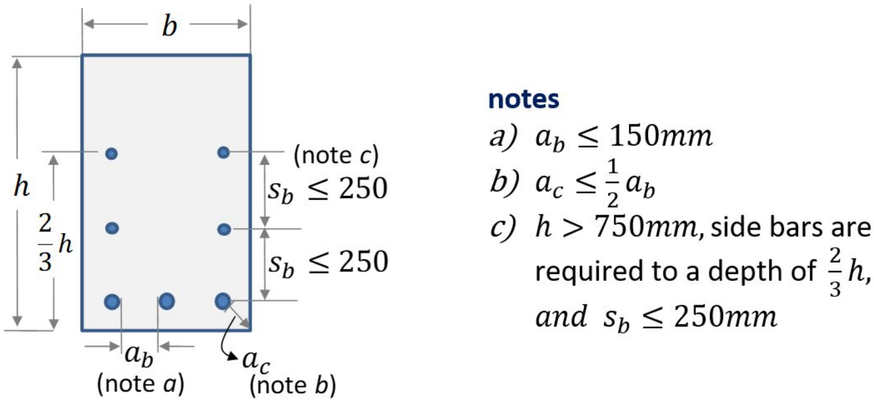

Figure 5.6: Crack control by limiting reinforcing bar spacing.

Side face reinforcement in beams for cracking control

For large beams ($h \geq 750\,\mathrm{mm}$) see Figure 5.6:

- Provide side reinforcement in the bottom $\frac{2}{3}h$ to control cracking.
- Spacing between side bars $s_b \leq 250\,\mathrm{mm}$.
- Bar diameter $\phi = 0.75\sqrt{b_w}$ for high-yield steel ($f_y = 500\,\mathrm{MPa}$), where $b_w \leq 500\,\mathrm{mm}$.
- If torsion reinforcement is already present and spacing $s_v \leq 250\,\mathrm{mm}$, additional side reinforcement is not required.

5.2 Curtailment of Reinforcement Bars

Curtailment refers to reducing the length of steel reinforcement bars where the bending moment decreases, optimizing material use. In reinforced concrete beams, the bending moment varies along the span, with maximum values typically at midspan for simply supported beams or at supports and midspan for continuous beams. Since the maximum moment is localized, it is possible to reduce the reinforcement in areas where the moment is lower, thereby saving material. Hence, curtailment saves steel but must ensure remaining bars provide sufficient moment resistance. To achieve this, two methods can be employed:

1. **Capacity Checking**: This involves calculating the required reinforcement at various sections along the beam based on the local bending moment, essentially solving the design problem in reverse.
2. **Standard Rules**: Most design codes provide simplified rules for curtailing reinforcement, specifying where and how many bars can be terminated based on the beam's span and loading conditions.

The standard rules approach is more commonly used in practice due to its simplicity and the assurance that it meets the code requirements. Regardless, when curtailing bars, it is essential to ensure that the remaining reinforcement is sufficient to resist the applied moments and that the bars are properly anchored to develop their full strength. This includes satisfying development length requirements and adhering to the detailing provisions of the design code.

Simply Supported Beams (Figure 5.7) Maximum moment occurs at midspan, requiring full reinforcement. As moment drops toward supports, fewer bars suffice. Code permits curtailing half the (bottom) bars from midspan at 0.1L from the centre of supports, ensuring remaining steel handles the reduced load.

Continuous Beams (Figure 5.7) Negative moments arise at supports, positive ones at midspan. For bottom bars, only 30% of the bottom bars from midspan are needed at a distance 0.1L and 0.15L from the end and internal support, respectively. For top bars, only 20% and 60% of the bars at the support are needed after a distance 0.25L and 0.15L away from the face of the support.

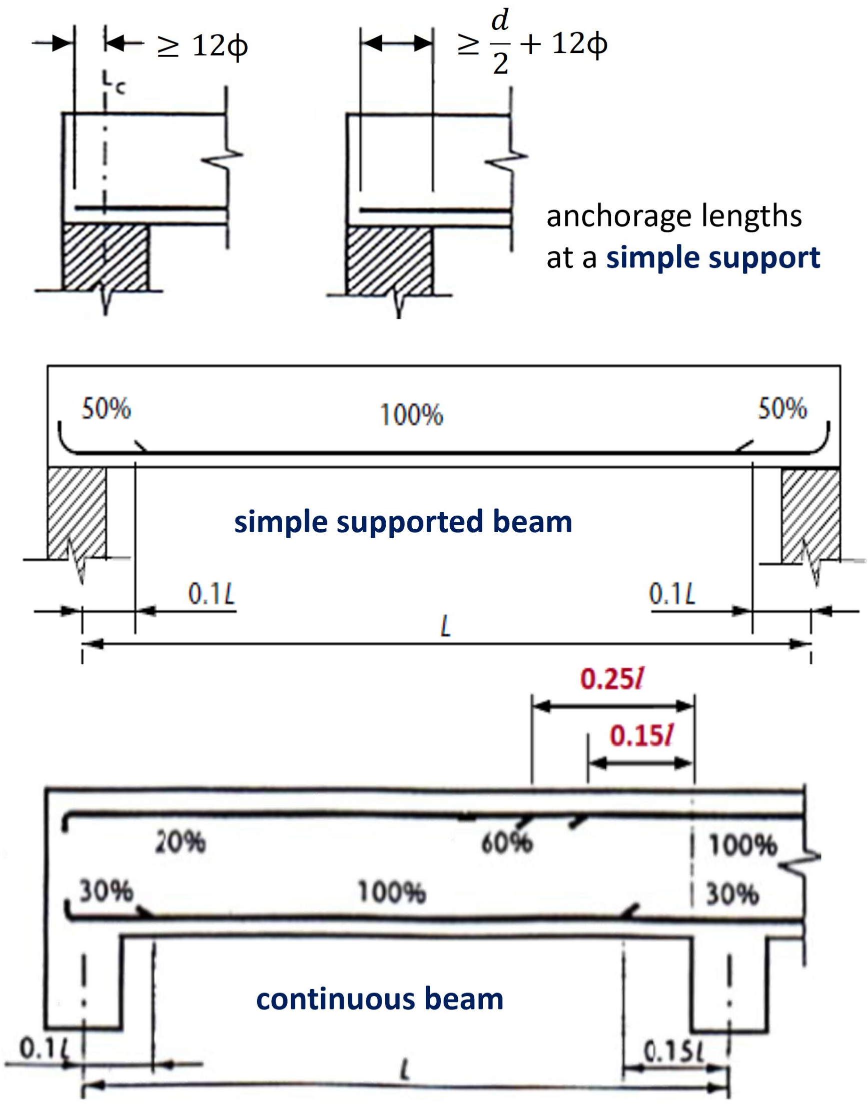

Figure 5.7: Curtailment rules for simply supported and continuous beams.

Chapter 6

Serviceability Limit States (SLS)

Overview: This section covers the following topics:

- Definition of limit states: Ultimate Limit States (ULS) for collapse prevention and Serviceability Limit States (SLS) for usability, alongside other states like durability and fire resistance.
- Key SLS considerations for RC structures: excessive deflection and crack width, with their impacts on appearance, function, and durability.
- Approaches to SLS design: deemed-to-satisfy provisions (e.g., span-to-depth ratios) and detailed analysis, with load combinations using a factor of 1.0.
- Detailing requirements for serviceability, including nominal concrete cover for durability and fire resistance, and bar spacing rules for cracking control.
- Deflection control methods: limiting span-to-effective depth ratios (L/d) and calculating deflection, with specific L/d values for various support conditions.
- Crack control strategies: deemed-to-satisfy bar spacing limits and crack width calculations, targeting a maximum of $0.3\,\mathrm{mm}$ under normal conditions.

6.1 Introduction to Serviceability Limit States

Limit states are conditions at which a structure becomes unfit for its intended function. In the context of this course we focus on two groups:

- **Ultimate limit states (ULS)**: states leading to collapse, e.g., ensuring the bending moment M does not exceed the ultimate moment capacity M_u.
- **Serviceability limit states (SLS)**: states disrupting the use of structures but not causing collapse.

In general, other limit states may concern:

- **Durability**: This must be considered in terms of the proposed life of a structure and its conditions of exposure. It involves ensuring that the structure can withstand environmental conditions (e.g., moisture, freeze-thaw cycles) without significant deterioration, achieved through appropriate concrete mixes, cover depths, and crack control.
- **Fire resistance**: This must be considered to prevent collapse, flame penetration, and excessive heat transfer during a fire. It is typically achieved by providing adequate concrete cover to reinforcement and using fire-resistant materials, ensuring the structure maintains its integrity for a specified duration (e.g., 1–2 hours).
- **Excessive undesirable vibration**: This can cause discomfort or alarm to occupants and may lead to damage. It is particularly important for floors in buildings like offices or hospitals, where excessive vibration can disrupt comfort or affect sensitive equipment (e.g., medical devices).
- **Fatigue**: This may be considered if cyclic loading is likely, such as in bridges or crane girders. It involves ensuring that the reinforcement does not fail under repeated stress cycles, which could lead to progressive cracking or fracture.
- **Special circumstances**: Any special requirements, such as explosion resistance (e.g., nuclear or thermal), or impact resistance, which demand specific design considerations beyond standard SLS and ULS checks.

For reinforced concrete (RC) structures, the two major serviceability limit states that normally must be considered in design are:

- **Excessive deflection**: The appearance or efficiency of a structure or any part of it must not be adversely affected by deflections.
- **Excessive crack width**: The appearance or durability of a structure or any part of it must not be adversely affected by any cracking of the concrete.

Excessive deflection can lead to visual sagging, which may be aesthetically displeasing, and can also cause functional problems such as malfunctioning of doors and windows or damage to finishes like plaster. To control deflection, design typically relies on span-to-depth ratio limitations or calculate the deflection and ensure it is within acceptable limits, such as span/250 for total deflection or span/360 for live load deflection, depending on the structure's use (e.g., floors versus roofs).

Excessive crack width can make the structure look unsightly and, more critically, can compromise durability by allowing water and aggressive agents (e.g., chlorides) to penetrate the concrete and corrode the reinforcement. Crack width is controlled by providing adequate reinforcement, limiting the stress in the reinforcement, and ensuring proper

concrete cover. Acceptable crack widths typically range up to $0.3\,\mathrm{mm}$, with stricter limits (e.g., $0.2\,\mathrm{mm}$) applied in aggressive environments to enhance durability.
Two approaches are commonly adopted for the design requirements of SLS:

- **Deemed-to-satisfy provisions**: These are rules and practices, such as detailing rules and span-to-depth ratio limitations, that are deemed to satisfy the SLS requirements without the need for detailed calculations. Examples include providing a minimum amount of reinforcement and maximum bar spacing for crack control (e.g., spacing not exceeding $200\,\mathrm{mm}$) or using span-to-depth ratios (e.g., 20 for simply supported beams) to limit deflection.

- **Analysis**: This involves calculating the effects of loads, such as deflections and crack widths, and comparing them with acceptable values. This approach is used when deemed-to-satisfy provisions are not applicable (e.g., unusual geometries) or when more precision is required, such as in complex or non-standard structures where detailed modeling is necessary.

In most cases of day-to-day practical design, SLS requirements are met by deemed-to-satisfy provisions, simplifying the process for typical structures like residential or office buildings.

When needed, the load combinations for serviceability limit states use a load factor of 1.0 on all service loads, reflecting the conditions under normal use. For example, the SLS load combination for dead load (G_k), imposed load (Q_k), and wind load (W_k) is:

$$1.0G_k + 1.0Q_k + 1.0W_k \tag{6.1}$$

This contrasts with the ULS load combination, which uses higher load factors to account for safety against collapse, such as:

$$1.2G_k + 1.2Q_k + 1.2W_k \tag{6.2}$$

In design practice, the design of reinforced concrete structures is primarily based on ULS requirements to ensure safety against collapse. However, serviceability behavior is considered as a secondary check to ensure satisfactory performance under working conditions, addressing issues like deflection, cracking, and vibration. It is worth noting that in certain situations, such as architecturally exposed structures (where aesthetics are critical) or floors with sensitive equipment (where vibration must be minimized), SLS requirements may become the governing factor in design, necessitating more stringent control of deflections, crack widths, or vibrations.

6.2 Detailing Requirements for Serviceability

The design code provides simple rules to ensure satisfactory durability and serviceability of a structure under normal conditions, addressing exposure conditions, fire resistance, and the quantity and spacing of steel reinforcement. Table 6.1 classifies the environmental exposure conditions from 1 (mild) to 5 (abrasive):

exposure condition	type of exposure
1	**mild:** internal concrete surfaces; external concrete surfaces protected from the effects of severe rain or cyclic wetting and drying e.g. concrete finish with mosaic tiles...
2	**moderate:** internal concrete surfaces exposed to high humidity e.g. bathrooms and kitchens external concrete surfaces exposed to severe rain or cyclic wetting and drying
3	**severe:** concrete surfaces exposed to sea water spray through airborne contact but not direct exposure (near the coast); concrete surfaces exposed to corrosive fumes
4	**very severe:** concrete surfaces frequently exposed to sea or flowing water with pH = 4.5 concrete in sea water tidal zone down to 1 m below lowest low water level
5	**abrasive:** concrete surfaces exposed to abrasive action machinery, metal tyred vehicles or water carrying solids

Table 6.1: Types of Exposure Conditions for Concrete Structures

6.2.1 Nominal and Actual Concrete Cover

Nominal cover is the distance from the external surface of the concrete to the first reinforcement bar (typically a shear link) as indicated on drawings (Figure 6.1). It is critical for:

- Protect the steel against corrosion,
- Protect the steel against fire,
- Provide sufficient concrete depth for safe transmission of bond forces.

The **actual cover** differ from the nominal slightly (up to minus $5\,\mathrm{mm}$):

$$\text{actual cover} \geq (\text{nominal cover} - 5\,\mathrm{mm})$$

to account for construction tolerances.

Table 6.2 provides nominal cover values and minimum concrete grade (f_{cu}), to meet durability requirements [HKCC2013 [7]: Table 4.2] for pertinent exposure conditions. Table 6.2 also suggests relevant concrete mixes for different exposure conditions and specifies minimum cement content for normal weight aggregate of nominal size of $20\,\mathrm{mm}$, with adjustments for other sizes given in Clause 4.2.5.4. The concrete cover must not be less than the nominal cover for the environmental exposure condition, plus an allowance

condition of exposure (clause 4.2.3)	nominal cover (mm)						
	C20/25	C30	C35	C40	C45	C50	≥C55
condition 1							
- slabs only	30	30	25	25	25	25	25
- other members	35	30	30	30	25	25	25
condition 2	–	40	35	35	30	30	30
condition 3	–	–	–	50	45	45	45
condition 4	–	–	–	–	–	55	50
condition 5 (see note 3)							
maximum free water/cement ratio	0.65	0.65	0.60	0.55	0.45	0.40	0.35
minimum cement content (kg/m^3)	290	290	290	300	340	380	380

Table 6.2: Nominal Cover Requirements for Different Exposure Conditions and Concrete Grades

for abrasion loss, while also considering fire protection (Clause 4.3) and safe transmission of bond forces (Clause 8.7). The concrete cover must not be less than the nominal cover for the environmental exposure condition, plus an allowance for abrasion loss, while also considering fire protection (Clause 4.3) and safe transmission of bond forces (Clause 8.7). For prestressed concrete, use a grade above C30 with a minimum cement content of $300\,\mathrm{kg/m^3}$.

For example, for $f_{cu} = 30\,\mathrm{MPa}$ concrete:

- Condition 1: Nominal cover = $30\,\mathrm{mm}$ and hence actual cover $\geq 25\,\mathrm{mm}$
- Condition 2: Nominal cover = $40\,\mathrm{mm}$ and hence actual cover $\geq 35\,\mathrm{mm}$.
- Conditions 3–5: Use a higher grade of concrete.

Cover also influences the effective depth d:

$$d = h - c - \phi_T - \frac{\phi_L}{2},$$

where h is the overall depth of the section, ϕ_T is the shear link diameter, and ϕ_L is the longitudinal bar diameter. Therefore, for an actual cover of $30\,\mathrm{mm}$, and assuming $\phi_T = 10\,\mathrm{mm}$ and $\phi_L = 20\,\mathrm{mm}$, the difference between overall depth and effective depth is $h - d \approx 30 + 10 + 20/2 = 50\,\mathrm{mm}$, which can increase to $70\,\mathrm{mm}$ in harsher conditions. the difference between overall depth and effective depth is $h - d \approx 30 + 10 + 20/2 = 50\,\mathrm{mm}$, which can increase to $70\,\mathrm{mm}$ in harsher conditions.

The nominal cover specified for durability may not suffice for fire protection in some cases. Where applicable, adjust the nominal cover per the *Code of Practice for Fire Safety in Buildings 2011* (Buildings Department, 2nd version, 2012). For fire resistance, refer to minimum dimensions and cover for beams in Table 6.3 and for stairs in Table 6.3 of the same code.

6.2.2 Bar Spacing Requirements

1. Maximum spacing for cracking control

- **for beams:** maximum clear spacing for tension bars in beams is $300\,\mathrm{mm}$ for mild steel or $600\,\mathrm{mm}$ for high-yield steel when a maximum crack width of $0.3\,\mathrm{mm}$ is acceptable

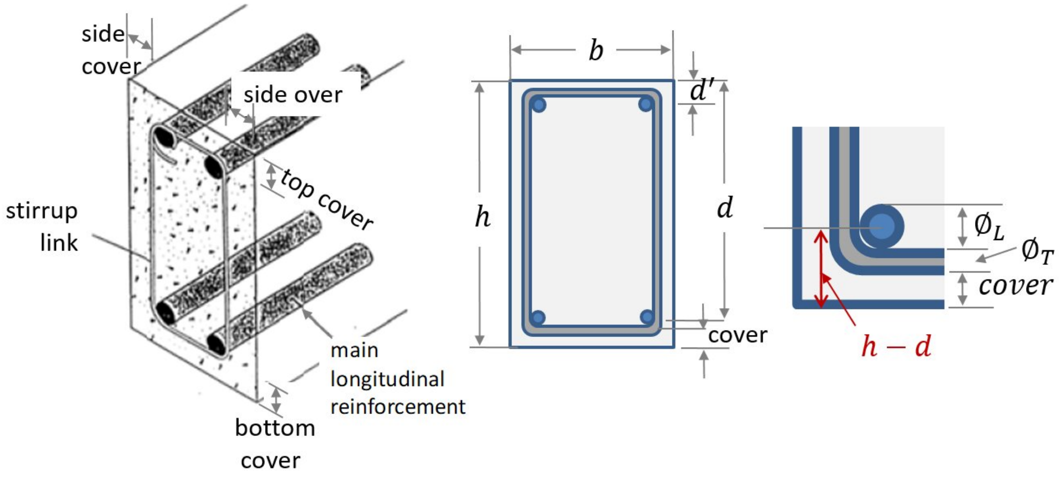

Figure 6.1: Concrete cover and difference between h and d .

	fire resistance rating (hours)		
	4	**2**	**1**
RC beams			
width of beam (mm)	280	200	200
width of beam (mm)	280	200	200
concrete cover to main reinforcement (mm)			
simply supported	80	50	30
continuous	60	40	30
RC stairs			
thickness at waist of slab (mm)	170	125	95
thickness at waist of slab (mm)	170	125	95
concrete cover to all reinf. (mm)	55	35	20

Table 6.3: Fire Resistance Ratings for RC Beams and Stairs

- **for slabs:** the maximum clear spacing between tension bars should not exceed $750\,\mathrm{mm}$ or $3d$ under the specified conditions $h \leq 200\,\mathrm{mm}$ with high-yield steel, or $\rho \leq 0.3\%$.

2. Minimum spacing for construction quality

Minimum spacing of reinforcements (Cl. 8.2 of the Code) clear distance (horizontal and vertical) is the greatest of:

- max bar diameter ϕ
- max aggregate size (h_{agg}) + $5\,\mathrm{mm}$
- $20\,\mathrm{mm}$

To permit concrete flow around reinforcement during casting, the minimum clear gap between bars, or group of bars, usually should exceed (Figure 6.2):

- ($h_{agg} + 5\,\mathrm{mm}$) horizontally and
- ($\frac{2h_{agg}}{3}$) vertically

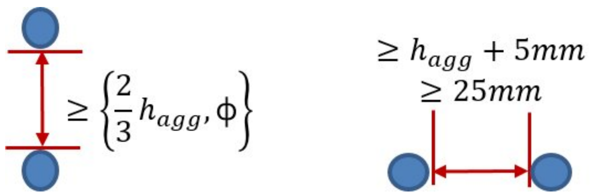

Figure 6.2: Minimum spacing for construction quality.

3. Maximum area of reinforcement for construction quality

Maximum areas of reinforcement are determined largely from the practical need to achieve adequate compaction of the concrete around reinforcement:

- **for beams:**
  1. the longitudinal reinforcement ratio,
     $\frac{A_s}{bh}$ or $\frac{A_s'}{bh} \leq 4\%$ each (clause 9.2.1.3)
     [for **ductility detailing**, $\frac{A_s}{bh} \leq 2.5\%$ (clause 9.9.1.2)]
  2. main bars in beams are normally ≥ 16 mm
  3. where bars are lapped, sum of the bar sizes in a layer must not be $> 40\%$ of the section breadth
- **for columns:**
  - $\frac{A_{sc}}{bh} \leq 6\%$ if cast vertically (4% in common use), 8% if cast horizontally, and 10% at laps if either case;
  - while $\frac{A_{sc}}{bh} \leq 4\%$ for ductility detailing (clause 9.9.2.1)
  - at least 4 bars with a diameter not less than 12 mm are required in a rectangular section and 6 in a circular section.

4. Minimum area for cracking control

- **for beams:** thermal and shrinkage cracking may be controlled by the use of minimum reinforcement quantities given in HKCC2013: Table 9.1 (see Table 6.4), while **detailing for ductility** $\rho_{min} = 0.3\%$ (clause 9.9.1.2)
- **for columns:** the minimum longitudinal reinforcement ratio $\frac{A_{sc}}{bh} = 0.8\%$ (clause 9.5.1), which is also suitable to detailing for ductility (clause 9.9.2.1)

Reinforcement Ratios

- **Maximum Reinforcement Ratio (ρ_{max}):** 4% without ductility requirements, 2.5% with ductility requirements (default for beams).
- **Minimum Reinforcement Ratio (ρ_{min}):** For high-yield steel ($f_y = 500\,\text{MPa}$), $\rho_{\text{min}} = 0.13\%$ without ductility, 0.3% with ductility. For flanged beams, use the web width b_w to calculate $\rho = \frac{A_s}{b_w h}$.

Minimum Bar Sizes Main bars in beams should have a diameter $\phi \geq 16\,\text{mm}$.

Situation	**Definition of percentage**	**Minimum percentage (%)** $f_y =$	
		250 N/mm^2	**500 N/mm**^{2}
Tension reinforcement subjected mainly to pure tension	$100\frac{A_s}{A_c}$	0.8	0.45
Sections subjected to flexure:			
(i) flanged beams, web in tension:			
$b_w/b < 0.4$	$100A_s/(b_w h)$	0.32	0.18
$b_w/b \geq 0.4$	$100A_s/(b_w h)$	0.24	0.13
(ii) flanged beams, flange in tension:			
T-beam	$100A_s/(b_w h)$	0.48	0.26
L-beam	$100A_s/(b_w h)$	0.36	0.20
(iii) rectangular section	$100A_s/A_c$	0.24	0.13
Compression reinforcement (where such reinforcement is required for the ultimate limit state)			
General rule	$100A_{sc}/A_{cc}$	0.4	0.4
Simplified rules for particular cases:			
(i) rectangular beam	$100A_{sc}/A_c$	0.2	0.2
(ii) flanged beam			
flange in compression	$100A_{sc}/(bh_f)$	0.4	0.4
web in compression	$100A_{sc}/(b_w h)$	0.2	0.2
Transverse reinforcement in flanges of flanged beams (provided over full effective flange width near top surface to resist horizontal shear)	$100A_{st}/h_f l$	0.15	0.15

Table 6.4: Minimum percentage of reinforcement for beams(HKCC2013: Table 9.1)

6.2.3 Deflections

Deflection control ensures that structural elements, such as beams and slabs, do not deform excessively under load. Excessive deflections may lead to:

- sagging floors, and to roofs that do not drain properly
- damaged partitions and finishes, and to other associated troubles

In general, the final deflection (including the effects of creep and shrinkage) of a beam, slab or cantilever should not exceed span/250 [HKCC2013: clause 7.3.1] [3, 16]

Excessive response to wind load

- Excessive accelerations that may cause discomfort or alarm to occupants should be avoided. Accordingly, the maximum peak acceleration of buildings is assessed with the following limits (clause 7.3.2):
  - for residential buildings, $a_{peak} = 0.15\,\mathrm{m/s^2}$
  - for office or hotel buildings: $a_{peak} = 0.25\,\mathrm{m/s^2}$
- The top drift of a building should $\leq H/500$, where H = the building height (clause 7.3.2)
- storey drift (relative lateral deflection in any one storey), which may damage to non-structural elements, should $\leq h/500$, where h = the storey height [BS 8110: Part 2: clause 3.2.2.2]

Methods for deflection control

Two primary methods are used:

- **Limit the span-to-effective depth ratio (L/d)**: This widely adopted method suits most practical cases.
- **Calculate deflection from curvature**: This involves analyzing the cross-section under applied moments but is less common in routine design.

In practice, designers typically control deflections by limiting the L/d ratio rather than performing detailed calculations.

Basic Span-to-Effective Depth Ratios for Beams and Slabs

Design codes provide basic L/d ratios for rectangular and flanged sections to limit total deflection to span/250. The L/d limits vary by structural system (see Table **??**). If the L/d ratio exceeds the limit, we can increase the depth h to reduce L/d. This approach simplifies the complex task of calculating stiffness in reinforced concrete, which is challenging due to cracking and material variability. Additionally, the deflection after construction of finishes and partitions should not exceed $\min\{\text{span}/500, 20\,\mathrm{mm}\}$ for spans up to 10 m. These basic ratios must be adjusted based on the actual span and reinforcement provided.

exposure condition	reinforced/prestressed members with	
	unbonded tendons quasi-perm. load	**bonded tendons freq. load**
1, 2, and 3	0.3 mm[(1)]	0.2 mm
4	0.3 mm	0.2 mm

Table 6.5: Crack Widths for Different Exposure Conditions

Notes

1. The given L/d values are generally conservative; calculations may allow shallower sections.
2. For two-way spanning slabs continuous over one long side, an L/d of 23 is suitable.
3. For two-way spanning slabs, use the shorter span for deflection checks.
4. Typical beam span/depth ratios in design practice are as follows: simply supported: $L/d = 10 - 12$, continuous: $L/d = 12 - 15$, and cantilever: $L/d = 5 - 6$

6.2.4 Cracking

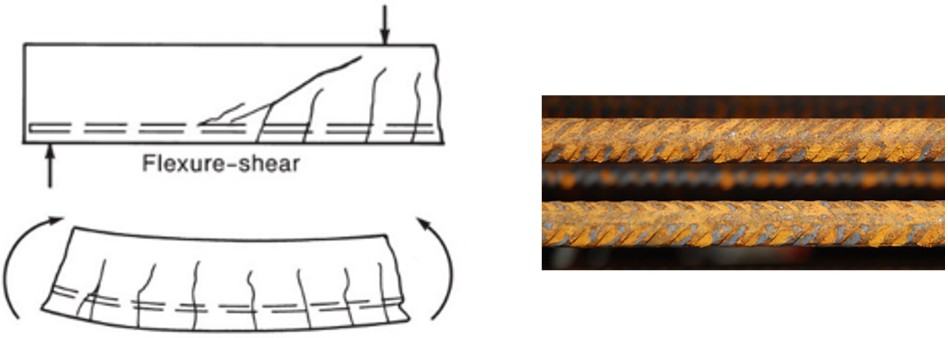

Figure 6.3: Cracking in R/C and corrosion.

Excessive cracking in reinforced concrete can form wide, deep cracks that compromise durability and lead to reinforcement corrosion (eg Figure 6.3).

Crack Control in Reinforced Concrete

According to HKCC2013 (Clause 7.2.1, Table 7.1), maximum surface crack widths are limited to ensure satisfactory appearance and durability in RC members of buildings [3, 16]. Under normal exposure conditions, the maximum allowable crack width is 0.3 mm (see Table 6.6).

Cracks in reinforced concrete are inevitable but must be controlled to $w \leq 0.3\,\mathrm{mm}$ (for reference, a human hair is $\leq 0.1\,\mathrm{mm}$). Two control methods are available:

1. **Deemed-to-Satisfy Provisions**: Limit bar spacing in the tension zone (see Section 3.2).
2. **Calculate Crack Width**: Calculate the crack width using the formula in HKCC2013: Clause 7.2.3 (or in BS 8110: Part 2: Section 3.8 for special cases) and compare it against limits (not covered in this course).

Again, the common approach is the first method: deemed-to-satisfy provisions.

exposure condition	**reinforced/prestressed members with**	
	unbonded tendons quasi-perm. load	**bonded tendons freq. load**
1, 2, and 3	0.3 mm$^{(1)}$	0.2 mm
4	0.3 mm	0.2 mm
water retaining structures$^{(2)}$	0.2 mm	–

Table 6.6: Maximum Crack Widths for Different Exposure Conditions

FAQs on Serviceability and Detailing

This is a collection of frequently asked questions (FAQs) related to Detailing and Serviceability Limit State (SLS) design, based on common student inquiries and clarifications provided during lectures.

Q1: What is the meaning of "Design for Serviceability Limit State (SLS)"?

SLS design (covered in Lecture 13) focuses on ensuring the structure remains functional and comfortable during its intended life. This primarily involves limiting **deflections** and **crack widths** to acceptable levels. In short, design should consider and satisfy all requirements (ULS and SLS).

Q2: What if spacing requirements for SLS contradict other design parameters (e.g., cover or section size)?

If clear spacing limits are not met, you can rearrange reinforcement into multiple layers. Ensure the new arrangement still provides the required A_s and maintains the effective depth d. Alternatively, use a larger number of smaller diameter bars to reach the same A_s while improving spacing and bond.

Q3: Which detailing components (anchorage, laps, bends) are required in sketches?

You only need to perform explicit calculations for anchorage, laps, and bends if specifically requested in the problem statement. However, arranging bars in layers and ensuring they fit within the beam width is a core part of any design sketch.

Q4: Why is a 50 mm cover sometimes used if the HK Code says 30 mm?

See Section **?????**.

Q5: How do I determine stirrup dimensions if the cover is unknown?

The concrete cover is a design choice, not an unknown. You must first select a cover based on code requirements (e.g., 30, 35, or 40 mm) before calculating the stirrup width and height.

Q6: Will exam questions require sketching flexural reinforcement?

Yes. Since shear and torsion design depend directly on the flexural design (e.g., the amount and location of tension steel), you must be able to sketch and interpret the full reinforcement layout.

Q7: What are "hang-up bars" and are they the same as longitudinal bars?

Yes, they are longitudinal bars. In a doubly reinforced beam, the compression reinforcement (A'_s) serves as the hang-up bars for the stirrups. In a singly reinforced beam, you must provide extra nominal longitudinal bars at the top to anchor and hold the stirrups in position.

Q8: Are there specific requirements for using 4-legged stirrups?

Yes. To properly anchor and maintain the shape of 4-legged stirrups, you should have at least **three bars** along both the top and bottom faces. While symmetry is preferred, it is not strictly mandatory to have the same number of bars on both sides.

Q9: Is the shear reinforcement different at column-beam joints?

Yes. Detailing at joints requires special consideration see **Lecture ???**.

Chapter 7

Slabs

Overview: This chapter covers the following topics:

- Classification of slabs based on spanning and load distribution: cantilever, one-way, and two-way slabs.
- Analysis of one-way slabs, including load transfer along the short span and calculation of uniformly distributed loads (UDL).
- Analysis of two-way slabs, including load distribution along both spans, tributary areas, and use of shear force coefficients.
- Design considerations for slabs, such as preliminary sizing, span/depth ratios, and load combinations.
- Reinforcement detailing for one-way and two-way slabs, covering moment steel, distribution steel, and torsion reinforcement.
- Punching shear failure in slabs, its analysis, and design examples for ultimate punching load and required slab thickness.

7.1 Intro

Slabs are planar (usually) horizontal structural elements with thickness much smaller ($<$ $1/10$) than their other two dimensions and with primary loading perpendicular to their plane. Reinforced concrete slabs are widely used as floors, roofs, and foundations of buildings, as well as decks of bridges, due to their versatility and strength. Based on the construction method, slabs can be categorized as cast in situ or precast. Slabs are typically supported by concrete beams or, less commonly, steel beams (Figure 7.1). They can also be designed to rest directly on columns, as seen in flat slabs, or variations with thickened sections near columns for enhanced shear capacity as in waffle slab, or slabs with drops. Additionally, slabs can be constructed with ribbed or voided cross-sections to improved strength-to-weight ratios and optimize material use.

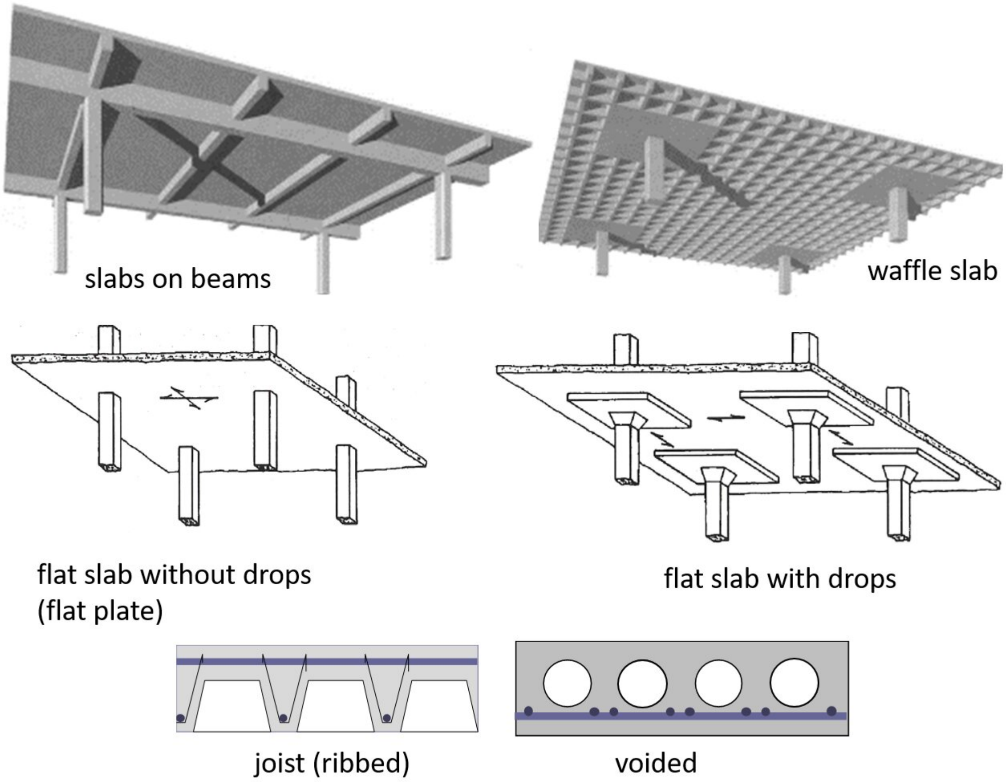

Figure 7.1: Types of slabs

7.2 Load Transfer - Principal Slab Types

Slabs are primarily classified based on how they span and distribute loads. For the purpose of this course the main types are:

- cantilever slabs
- one-way slabs
- two-way slabs

To introduce the different types of slabs let us classify slabs S1 to S4 of Figure 7.2, and calculate how they distribute their load. Assume the load per area is

$$n \rightarrow \mathrm{kN/m^2}$$

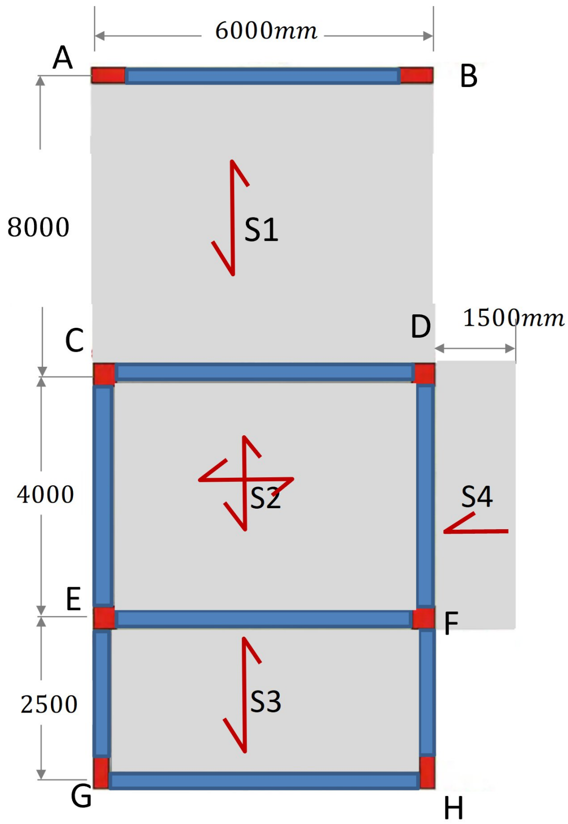

Figure 7.2: A floor plan with four slabs.

7.2.1 Cantilever

Cantilever slabs are widely utilized in balconies, overhangs, canopies, or architectural features where an unsupported projection is desired. These slabs are supported only at one edge, functioning as a cantilever with a fixed end and a free end. All loads acting on the cantilever slab are transferred to the fixed support. In the floor plan shown in Figure 7.2, slab S4 is a cantilever slab.

For a uniformly distributed area load n (in $\mathrm{kN/m^2}$), the uniformly distributed load (UDL) per unit length along the supporting edge beam DF is (Figure 7.3):

$$q = n \cdot l_x$$

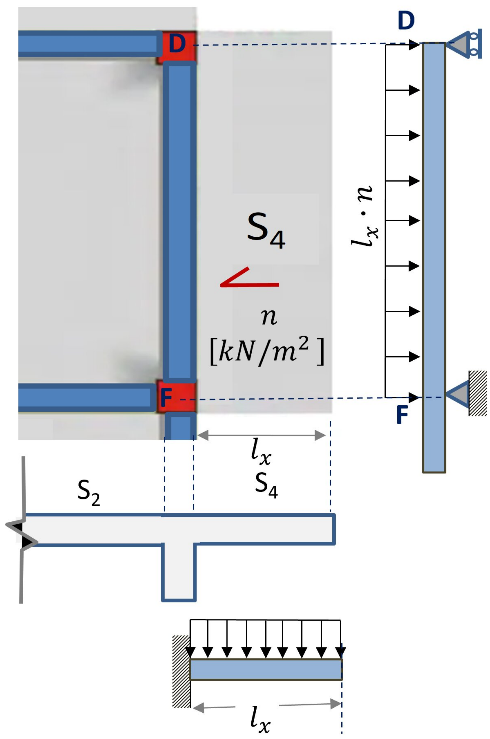

Figure 7.3: Slab S4 is a cantilever slab in Figure 7.2. The red arrow indicates the spanning direction.

where $l_x = 1500\,\mathrm{mm}$ represents the span of the cantilever.

Similar to cantilever beams, for gravity loads, the moment in a cantilever slab is negative, inducing tension on the top surface and compression on the bottom. Consequently, steel reinforcement is placed in the top layer of the slab, extending from the support into the slab. To determine the moments in a cantilever slab, we analyze a 1-meter-wide strip as a beam. The maximum bending moment of the catilever slab S4 at the support is given by:

$$M = \frac{n \cdot l_x^2}{2} = \frac{n \cdot (1.5)^2}{2} \quad \mathrm{kN\,m/m}$$

7.2.2 One-Way Slabs

One-way slabs transmit loads in a single direction to their supporting edges. This behavior occurs under the following conditions:

- When a solid slab is supported only on two opposite sides, it functions as a one-way slab, spanning perpendicular to those sides, regardless of the slab's side ratio. For instance, in the floor plan of Figure 7.2, slab S1 is a one-way slab. Notably, S1 is spanning along the longer span since the only supports are along the perpendicular shorter direction.
- When a solid slab is supported on all edges and the ratio of its sides exceeds 2

$$\frac{l_y}{l_x} > 2,$$

  it behaves as a one-way slab spanning along the shorter span. Here:

  - $l_y = l_{max}$ = long span,

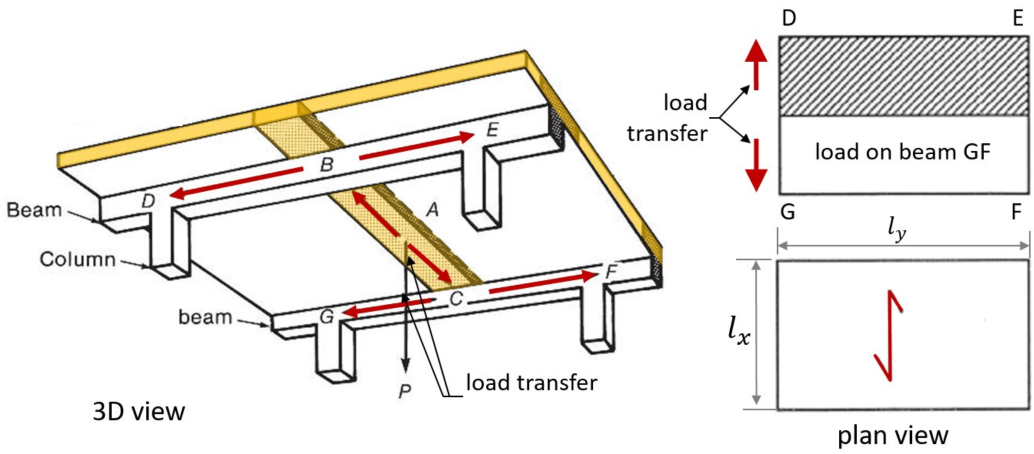

Figure 7.4: One-way slabs transfer all loads along the short span l_x.

- $l_x = l_{\min}$ = short span.

In the floor plan of Figure 7.2, slab S3 exemplifies this case, as it is supported on all edges with $l_y/l_x = 6/2.5 > 2$.

In both scenarios, one-way slabs transfer their loads along a single direction. Consequently, bending is considered only along that direction. When the slab is supported on all edges that direction is the direction of the short span, as illustrated in Figure 7.4. For one-way slabs S1 and S3 in Figure 7.5, the load per unit length transferred to the supporting beams is calculated as:

$$\text{UDL} = q = \frac{1}{2} l_x \cdot n \quad \mathrm{kN/m}$$

where n is the uniformly distributed area load, and half of this load is distributed to each supporting beam.

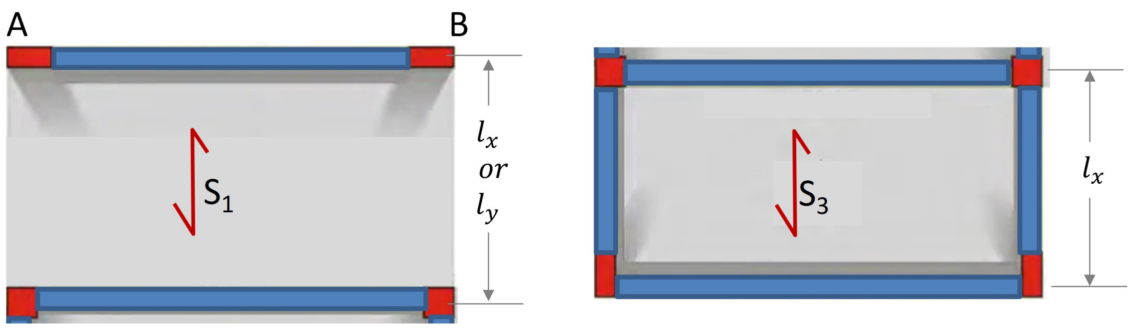

Figure 7.5: Slabs S1 and S3 are one-way slabs in Figure 7.2. The red arrows indicate the spanning direction, which aligns with the direction of load transfer.

The physical mechanism behind load transfer/distribution is governed by **stiffness**. At any a given point of the slab, the short and long spans deflect by the same amount. For example in the slab of Figure 7.7 $w_x = w_y$. This implies that forces in each direction (p_x and p_y respectively in Figure 7.7) are proportional to stiffness, with stiffer directions resisting deflection with greater force. Assuming that the slab is of uniform thickness

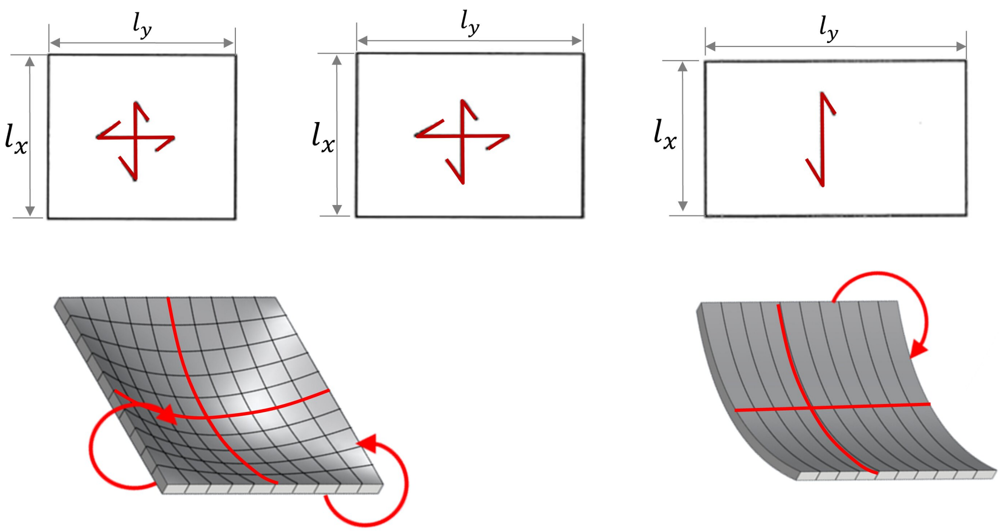

Figure 7.6: Depending on the span ratio a slab supporting on all edges can be a two-way or a one-way slab. The red arrows indicate the spanning directions.

and material properties the flexural rigidity EI is constant (where E is the modulus of elasticity and I the moment of inertia) and hence stiffness k is proportional to $\frac{1}{L^3}$, where L is the span length. Subsequently, in a slab where short span l_x and long span l_y ($l_y > l_x$) have same support conditions (e.g., simple or fixed supports), the stiffness in the short span direction is $k_x \propto \frac{1}{l_x^3}$, and in the long span direction, it is $k_y \propto \frac{1}{l_y^3}$. For example, if $l_y = 2l_x$, then $k_y \propto \frac{1}{(2l_x)^3} = \frac{1}{8l_x^3}$, indicating that under identical support conditions the short span is eight times stiffer (i.e., almost an order of magnitude greater). If the stiffness in one direction is an order of magnitude higher—due to shorter span length—the forces in that direction are correspondingly higher, as force relates to stiffness and deflection via $p = k \cdot w$, where w is the deflection. Loads thus distribute toward the stiffer direction of the shorter span, causing the slab to bend primarily along that direction. In one-way slabs, where the short span direction is significantly stiffer, the slab behaves like parallel beams spanning the short direction, transferring almost all of the load to the supports at its ends.

7.2.3 Two-Way Slabs

A two-way slab spans in both directions, transferring loads along both spans and exhibiting bending in both the l_x and l_y directions (see Figure 7.8). This behavior occurs when a solid slab is supported on all four edges and the ratio of its spans satisfies:

$$\frac{l_y}{l_x} \leq 2$$

where l_y is the long span and l_x is the short span. In the floor plan of Figure 7.2, slab S2 is a two-way slab, as it is supported on all edges with a span ratio of $l_y/l_x = 6/4 < 2$.

- A square two-way slab ($l_y = l_x$) distributes loads equally in both directions to all four edges when support conditions are symmetric, as dictated by symmetry.
- A rectangular two-way slab ($l_y > l_x$) transfers the majority of the load along the shorter, stiffer span.

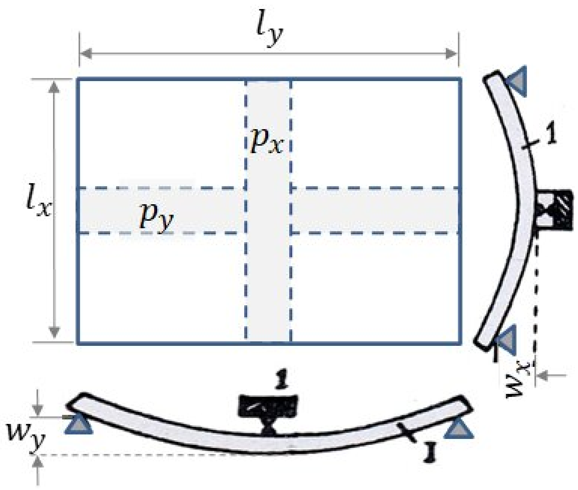

Figure 7.7: Short vs Long span strips and the corresponding forces and deflections.

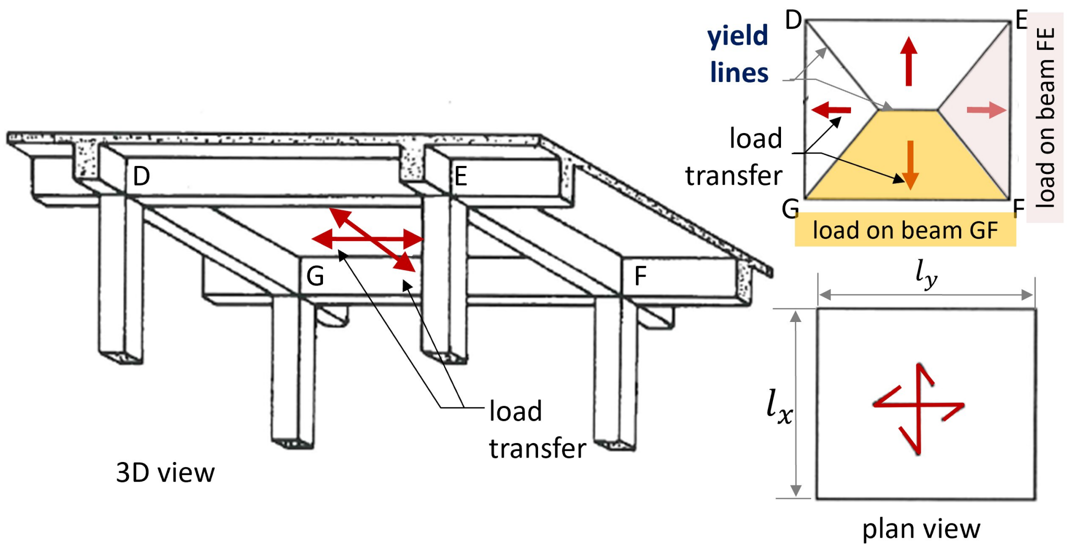

Figure 7.8: Two-way slabs transfer loads along both spans.

- If the span ratio $l_{max}/l_{min} > 2.0$ (see Figure 7.6), nearly all the load is carried along the short span, allowing the slab to be designed as a one-way slab spanning in that direction alone. A finite element analysis can demonstrate this.

The flow chart of Figure 7.9 explains the classification of a slab into cantilever, one-way, or two-way slab.

Distribution of Load on Supporting Edges of a Two-Way Slab

In general, two-way rectangular slabs distribute their area load to the supporting edges based on tributary areas defined by yield lines. Yield lines extend the concept of plastic hinges into two dimensions and can be approximated using patterns shown in Figure 7.10. The pattern of the yield lines (see e.g., Figure 7.24) determines the transfer of load to each edge. These patterns can be simplified using two **rules**:

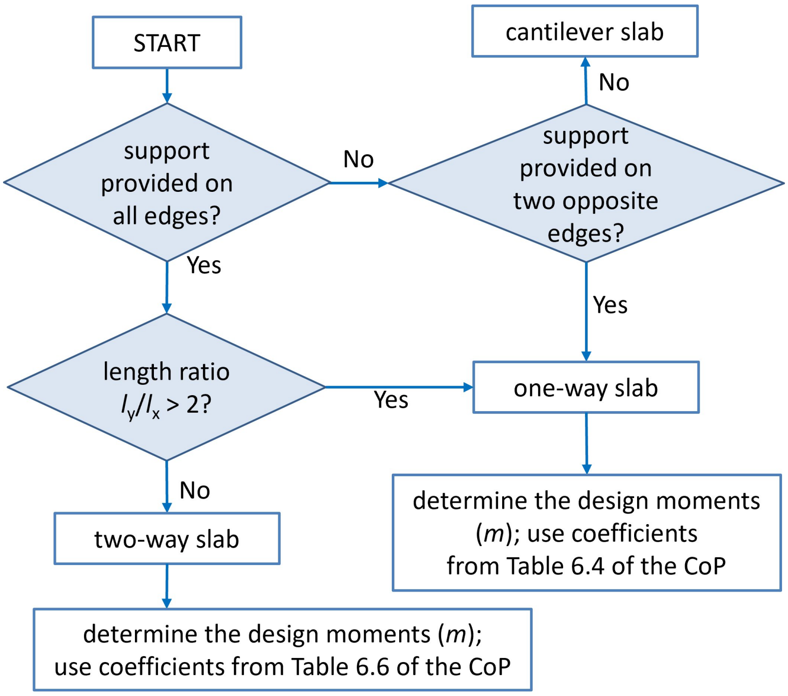

Figure 7.9: Slab classification flowchart.

- At corners with identical support conditions (e.g., simple-simple or fixed-fixed), the load distribution angle between the two directions is 45° (see Figure 7.10).
- At corners with mixed support conditions (e.g., one fixed and one simple), the distribution angles are 60° and 30°, respectively (see Figure 7.10).

The shape of the tributary areas results in a non-uniform load distribution—typically trapezoidal or triangular—along the supporting edges, see Figures 7.10 and 7.24. For practical design, these trapezoidal or triangular loads are often converted into equivalent uniformly distributed loads (UDL), as depicted on the right side of Figure 7.10.

A straightforward method to estimate the UDL on the supporting edges of a two-way slab is through shear force coefficients, provided in Table 7.1. Note that the shear force coefficients in Table 7.1 represent the shear forces within the slab itself, not directly on the supporting edges. To calculate the load on the supporting edges, we account for the fact that shear forces along the x-direction of the slab load the edges parallel to the y-direction, and vice versa:

- $v_s = v_{sx}$ when $l = l_y$,
- $v_s = v_{sy}$ when $l = l_x$.

Also, the shear force coefficients in Table 7.1 give the total force of the slab assuming this force is applied only in the middle $0.75 \cdot l$ of the span (i.e., the middle strip). Consequently, to distribute the same load to the whole span we multiply the shear force by 0.75 (see Figure 7.13 bottom).

Example: Slab with 4 Edges Discontinuous From Table 7.1 (Table 6.7 [HKCC2013, Clause 6.1.3.3]) for a slab with 4 edges discontinuous (i.e., simple supports) (Figure 7.13):

- span ratio $\alpha = l_y/l_x = 1.5$

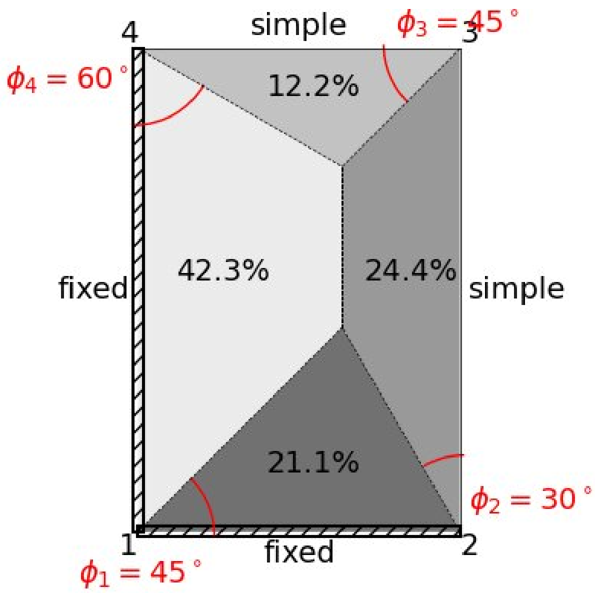

Figure 7.10: Tributary areas for load transfer in a two-way slab vary with support conditions.

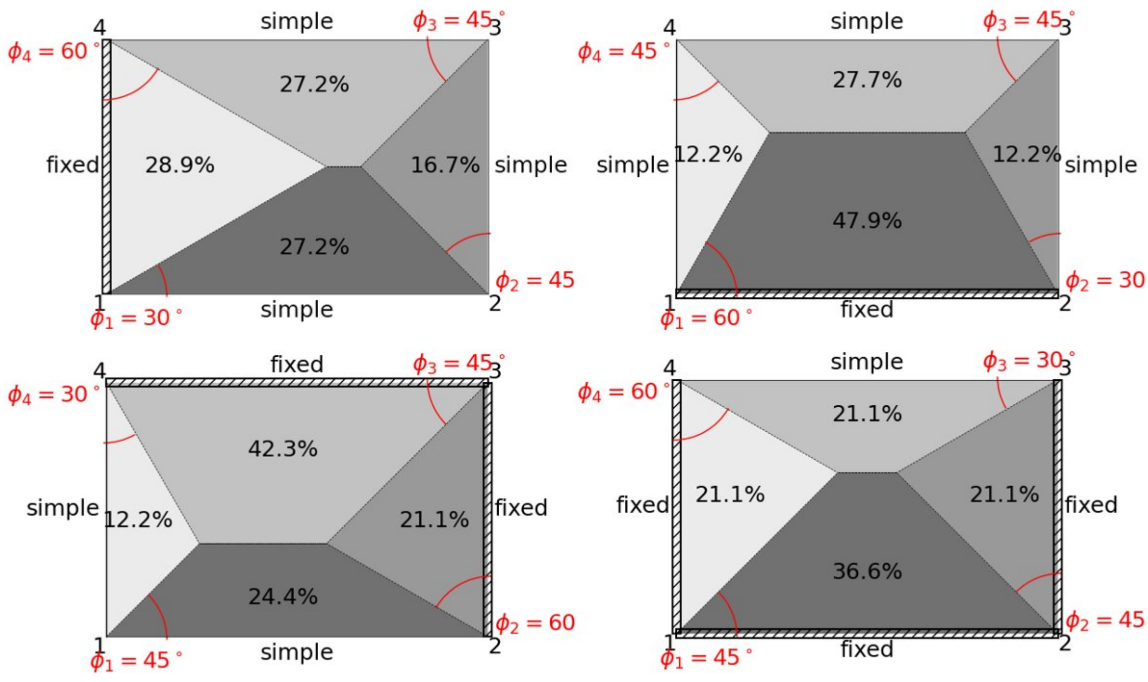

Figure 7.11: Tributary areas for load transfer in two-way slabs with $l_y/l_x = 1.5$ but different support conditions.

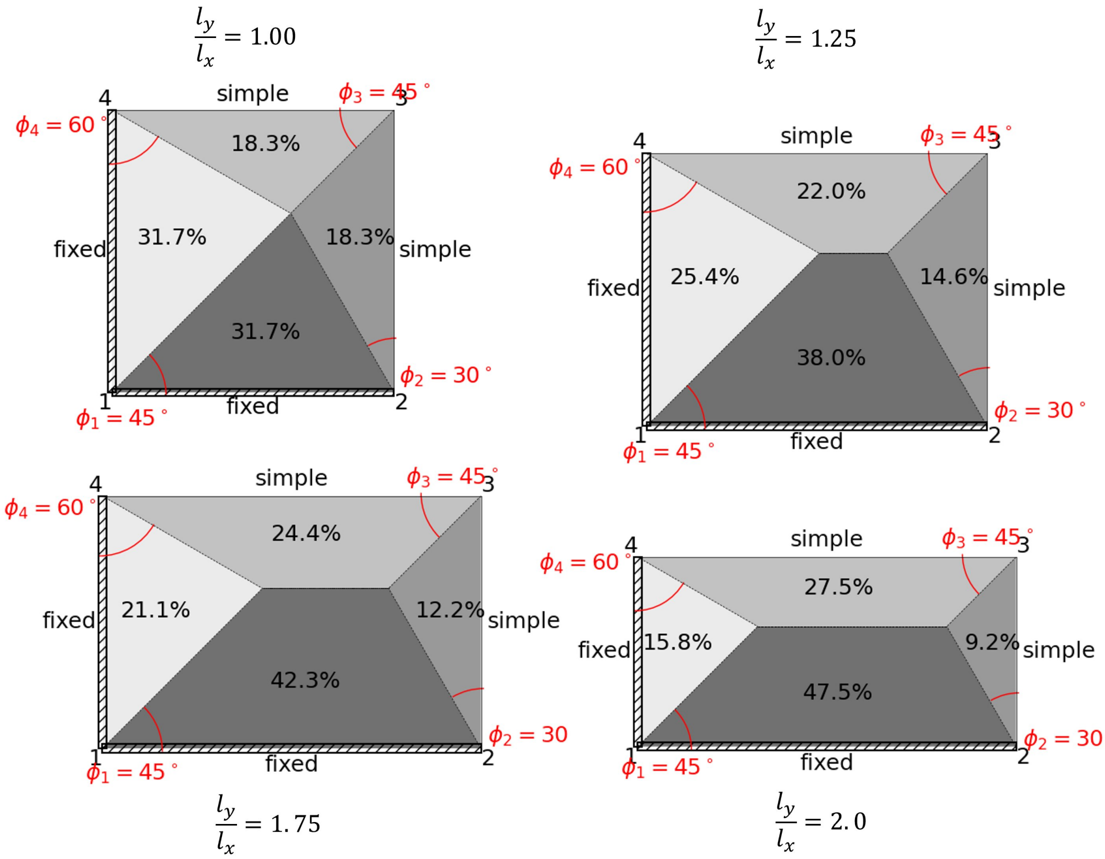

Figure 7.12: Tributary areas for load transfer in two-way slabs with same support conditions but different span ratios $\alpha = l_y/l_x$. Span ratio is $\alpha = 1.0$ for the top left slab, $\alpha = 1.25$ top right, $\alpha = 1.5$ bottom left, $\alpha = 2.0$ bottom right.

Table 7.1: Shear force coefficients for uniformly rectangular panels supported on 4 sides with provision for torsion at corners [HKCC2013 [7]: Table 6.7].

types of panel	location	short span coefficient β_{xx} for l_y/l_x								long span β_{vy}
		1.0	1.1	1.2	1.3	1.4	1.5	1.75	2.0	
4 edges	cont edge	0.33	0.36	0.39	0.41	0.43	0.45	0.48	0.50	0.33
1 short edge discontinuous	cont edge	0.36	0.39	0.42	0.44	0.45	0.47	0.50	0.52	0.36
	discnt edge	–	–	–	–	–	–	–	–	0.24
1 long edge discontinuous	cont edge	0.36	0.40	0.44	0.47	0.49	0.51	0.56	0.59	0.36
	discnt edge	0.24	0.27	0.29	0.31	0.32	0.34	0.36	0.38	–
2 adjacent edges discontinuous	cont edge	0.40	0.44	0.47	0.50	0.52	0.54	0.57	0.60	0.40
	discnt edge	0.26	0.29	0.31	0.33	0.34	0.35	0.38	0.40	0.26
2 short edges discontinuous	cont edge	0.40	0.43	0.45	0.47	0.48	0.49	0.52	0.54	–
	discnt edge	–	–	–	–	–	–	–	–	0.26
2 long edges discnt	cont edge	–	–	–	–	–	–	–	–	0.40
	discnt edge	0.26	0.30	0.33	0.36	0.38	0.40	0.44	0.47	–
3 edges discnt, 1 long edge cnt	cont edge	0.45	0.48	0.51	0.53	0.55	0.57	0.60	0.63	–
	discnt edge	0.30	0.32	0.34	0.35	0.36	0.37	0.39	0.41	0.29
3 edges discnt, 1 short edge cnt	cont edge	–	–	–	–	–	–	–	–	0.45
	discnt edge	0.29	0.33	0.36	0.38	0.40	0.42	0.45	0.48	0.30
4 edges discnt	discnt edge	0.33	0.36	0.39	0.41	0.43	0.45	0.48	0.50	0.33

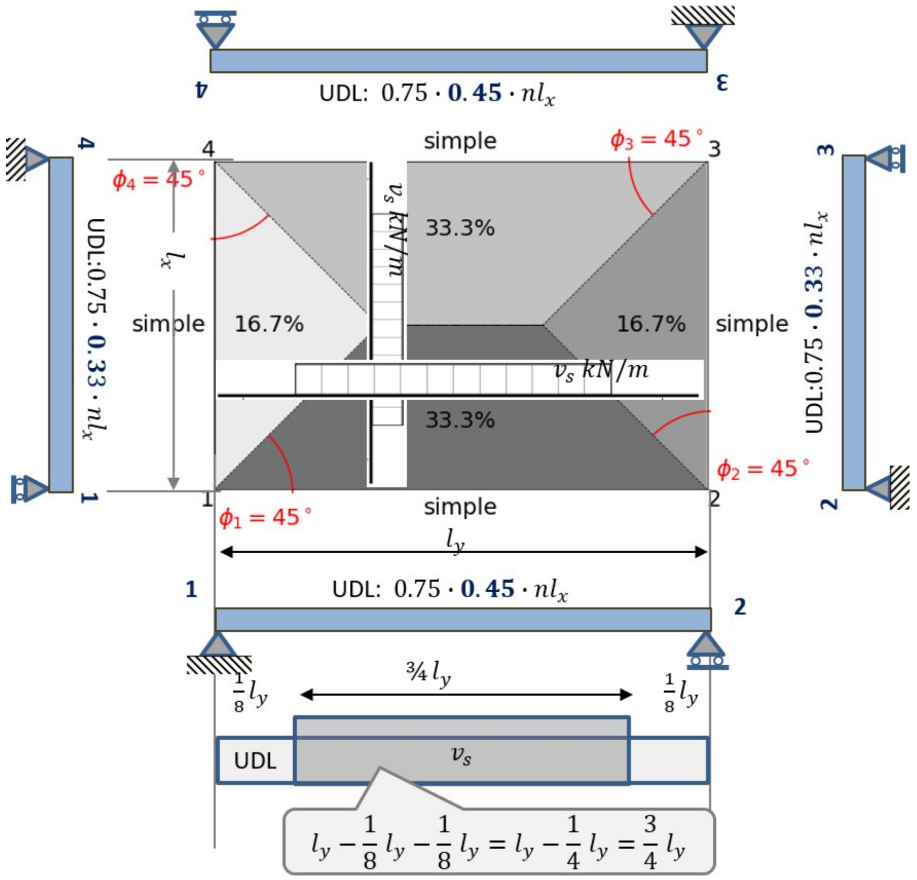

Figure 7.13: Rectangular slab with span ratio 1.5, simply supported on all edges: Load tributary areas and UDL on all supporting edges.

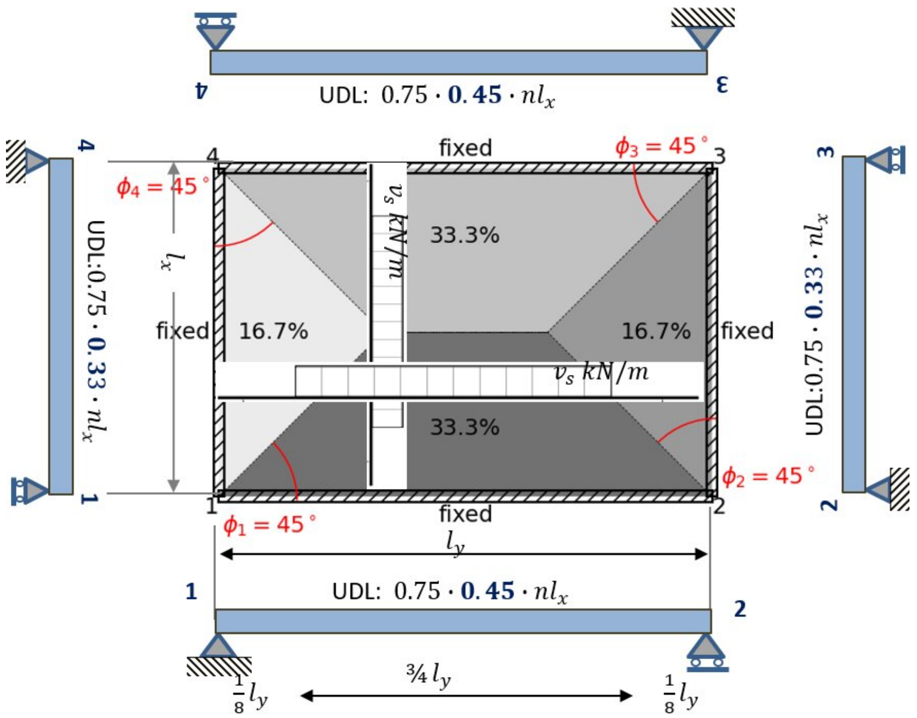

Figure 7.14: Rectangular slab with span ratio 1.5, continuously supported on all edges: Load tributary areas and UDL on all supporting edges.

- the UDL on supports AB & CD is:

$$0.75\beta_{vx}nl_x = 0.75 \cdot 0.45 \cdot nl_x$$

  where, β_{vx} is the shear coefficient, $n =$ the design load per area, and the coefficient 0.75 ensures that the same total load as calculated from Table 7.1 for the middle strip length $(0.75 \times l)$, is applied as UDL along the whole length of the supporting edge.

- the UDL on supports AD & BC is:

$$0.75\beta_{vy}nl_x = 0.75 \cdot 0.33 \cdot nl_x$$

  where, β_{vy} is the shear coefficient, and $n =$ the design load per area.

As a sanity check, we verify that the total load applied on the supporting beams equals the total load of the slab.

- The total load on a slab with span ratio $\alpha = l_y/l_x = 1.5$ is:

$$nl_x l_y = \alpha n l_x^2 = 1.5 n l_x^2$$

- The total load applied on the four supporting beams is:

$$0.75nl_x\left(0.33 \cdot l_x + 0.33 \cdot l_x + 0.45 \cdot \alpha l_x + 0.45 \cdot \alpha l_x\right) = 0.75nl_x^2 \cdot 1.74 = 1.5nl_x^2$$

Confirming that the total load on the slab is equal with the total load on the supporting edges.

For a slab with four edges discontinuous (i.e., simply supported) with a different span ratio $\alpha = l_y/l_x = 1.2$, from Table 7.1:

- The UDL on supports AB and CD is:

$$0.75\beta_{vx}nl_x = 0.75 \cdot 0.39 \cdot nl_x$$

  where β_{vx} is the shear coefficient, and n is the design load per unit area.

- The UDL on supports AD and BC is:

$$0.75\beta_{vy}nl_x = 0.75 \cdot 0.33 \cdot nl_x$$

  where β_{vy} is the shear coefficient, and n is the design load per unit area.

Example: Slab with 4 Edges Continuous

From Table 7.1 (Table 6.7 [HKCC2013, Clause 6.1.3.3]) for a slab with four edges continuous (i.e., fixed) (Figure 7.14):

- Span ratio $\alpha = l_y/l_x = 1.5$,

- The UDL on supports AB and CD is:

$$0.75\beta_{vx}nl_x = 0.75 \cdot 0.45 \cdot nl_x$$

  where β_{vx} is the shear coefficient, and n is the design load per unit area.

- The UDL on supports AD and BC is:

$$0.75\beta_{vy}nl_x = 0.75 \cdot 0.33 \cdot nl_x$$

  where β_{vy} is the shear coefficient, and n is the design load per unit area.

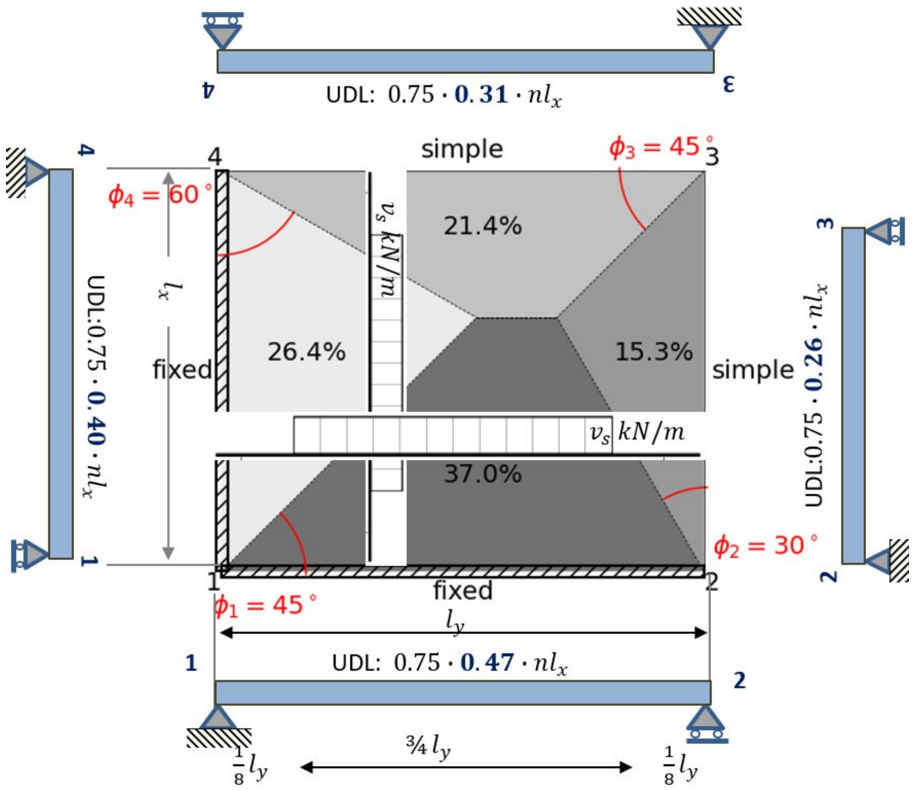

Figure 7.15: Rectangular slab with span ratio 1.2, two adjacent edges discontinuous: Load tributary areas and UDL on all supporting edges.

Example: Slab with 2 Adjacent Edges Discontinuous

From Table 7.1 (Table 6.7 [HKCC2013, Clause 6.1.3.3]) for a slab with two adjacent edges discontinuous (Figure 7.15):

- Span ratio $\alpha = l_y/l_x = 1.2$,
- The UDL on support AD is:

$$0.75\beta_{vy}nl_x = 0.75 \cdot 0.40 \cdot nl_x$$

  where β_{vy} is the shear coefficient, and n is the design load per unit area.
- The UDL on support BC is:

$$0.75\beta_{vy}nl_x = 0.75 \cdot 0.26 \cdot nl_x$$

  where β_{vy} is the shear coefficient, and n is the design load per unit area.
- The UDL on support CD is:

$$0.75\beta_{vx}nl_x = 0.75 \cdot 0.47 \cdot nl_x$$

  where β_{vx} is the shear coefficient, and n is the design load per unit area.
- The UDL on support AB is:

$$0.75\beta_{vx}nl_x = 0.75 \cdot 0.31 \cdot nl_x$$

  where β_{vx} is the shear coefficient, and n is the design load per unit area.

7.3 Design Considerations

7.3.1 Preliminary Sizing

Preliminary sizing is a critical step in slab design. As with beams, the effective depth has a major influence on the capacity of a slab to resist bending moment. Unlike beams though, a small increase in slab depth adds greatly to the self-weight of the whole structure. The increased self-weight, not only significantly increases the bending moments in the slab itself, but also adds load to all supporting beams, columns/structural walls, and foundations, increasing internal forces and thus leading to a significant increase in the construction cost.

In practice, for solid slabs, the following **guidelines** are often used:

- **Span/depth ratios** for the preliminary design of one-way spanning slabs are normally about the basic span/effective depth ratios for beams.
- Span/depth ratios of two-way spanning slabs are normally about 90% of the thickness of one-way spanning slabs.
- In common buildings, the **total depth** of spans ranges within $h = 120\,\mathrm{mm} \sim 200\,\mathrm{mm}$; normally take $h = 160\,\mathrm{mm}$
- The **effective depth** is usually $d = h - (30\,\mathrm{mm} \sim 35\,\mathrm{mm})$.

7.3.2 Overview of requirements

Reinforced concrete (R/C) slabs behave primarily as flexural members, and their design is similar to that for beams, though in general, it is somewhat simpler because:

- The slab breadth is fixed, and a unit value of 1 m cross-section width is normally used in calculations.
- Shear stresses are usually low, except where there are heavy concentrated loads.
- Compression reinforcement is seldom needed.

Load combinations/arrangements

When designing slabs, the single-load case of the maximum design load on all spans or panels can be considered:

$$1.4G_k + 1.6Q_k$$

provided that (Clause 6.1.3.2(c)) the following conditions are met:

1. In a one-way spanning slab the area of each bay exceeds $30m^2$.
2. The ratio of the characteristic imposed load to the characteristic dead load does not exceed 1.25; and
3. The characteristic imposed load does not exceed $5\,\mathrm{kN/m^2}$ excluding partitions.

The following adjustments are recommended:

- All support moments (except at a cantilever) should be reduced 20% to account for 20% moment redistribution ($\beta_b = 0.8$) and span moments increased accordingly.

- The pertinent provisions of EC2 for slab actions in building structures are very similar: EC2 also allows a simplified arrangement $\rightarrow$ all spans can be assumed to carry both the permanent and variable loads, provided the following conditions are met:
  - $q_k \leq 5\,\mathrm{kN/m^2}$
  - $\frac{q_k}{g_k} \leq 1.25$

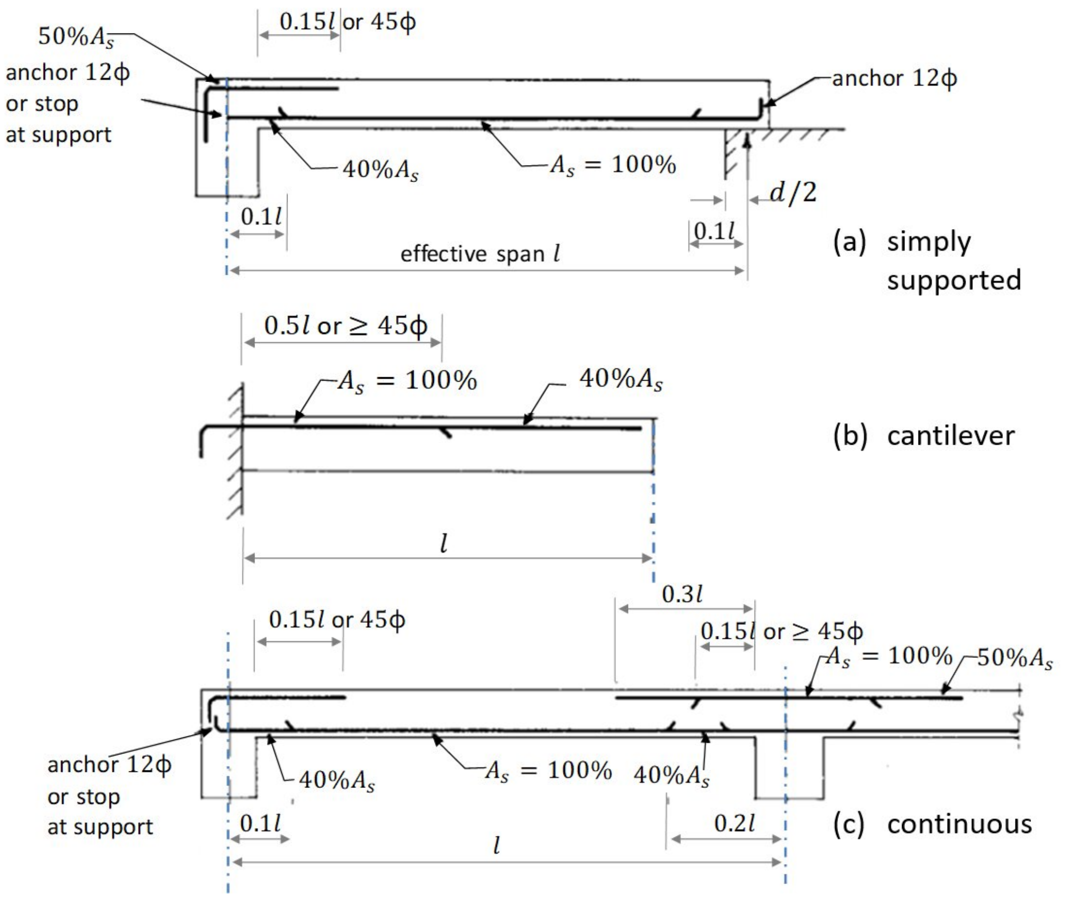

Figure 7.16: Curtailment and anchorage of reinforcement in slabs

Reinforcement detailing

Curtailment and Anchorage of Reinforcement in Slabs Reinforcement detailing is crucial in different structural configurations (Figure 7.16), such as: simply supported span, cantilever, or continuous beam slab.

7.4 One-Way Solid Slabs

One-way solid slabs are commonly used in construction due to their simplicity and efficiency in load distribution.

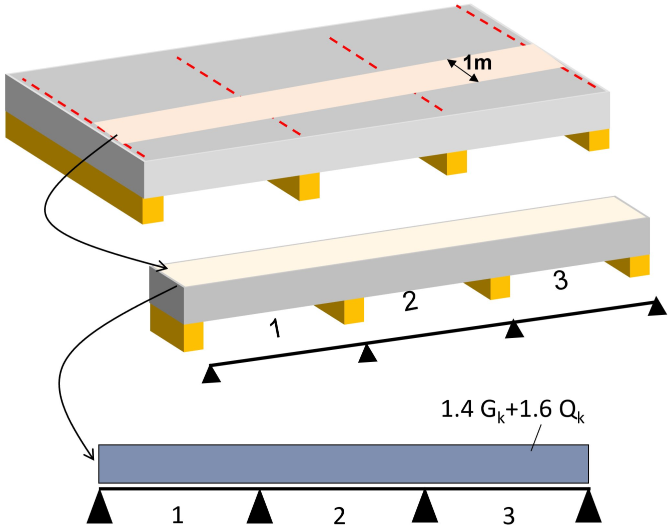

Figure 7.17: Structural system of a one-way slab, using a strip of 1m width.

Structural system of a one-way slab

Support Conditions One-way slabs can be simply supported (resting on beams without continuity) or continuous when for example the slab extends over the support. The structural system of a one-way slab can be thought as the structural system of a typical beam 1m wide along the spanning direction of the slab (see Figure 7.17). Once the structural system has been defined, we can proceed with structural analysis. Recall that instead of multiple load combinations in slabs we will use a single load combination with the maximum design load applied on all spans. For this single load case of maximum load on all spans, moment redistribution of 20 % should be considered. In other words, $\beta_b = 0.8$ should be to reduce support moments (except those at the supports of cantilevers) by 20 %, with a consequential increase in the span moments. Where a span is adjacent to a cantilever of significant length, the possibility should be considered of the case of slab unloaded/cantilever loaded.

Shortcut for structural analysis of slabs

For the single maximum load combination, the simplified formulas of Table (see Table 7.2) can be used to determine design bending moments and shear forces for continuous slabs instead of a rigorous structural analysis.
Notes on the calculation formulas of Table 7.2:

- F is the total design ultimate load ($1.4G_k + 1.6Q_k$) and L is the effective span:

$$F = (w = 1.4G_k + 1.6Q_k) \cdot L \rightarrow [\mathrm{kN/m}][\mathrm{m}] = [\mathrm{kN}]$$

- The coefficients of the table already account for 20% ($\beta_b = 0.8$) moment redistribution of support moments.

Table 7.2: Ultimate bending moment and shear forces in one-way spanning slabs

	simple end support		continuous end support				
	@ outer support	near middle of end span	@ outer support	near middle of end span	@ 1st interior support	middle interior spans	interior support
	e.g. A	e.g. AB	e.g. A	e.g. AB	e.g. B	e.g. C	
moment	0	0.086FL	-0.04FL	0.075FL	-0.086FL	0.063FL	-0.063FL
shear	0.4F	–	0.46F	–	0.6F	–	0.5F

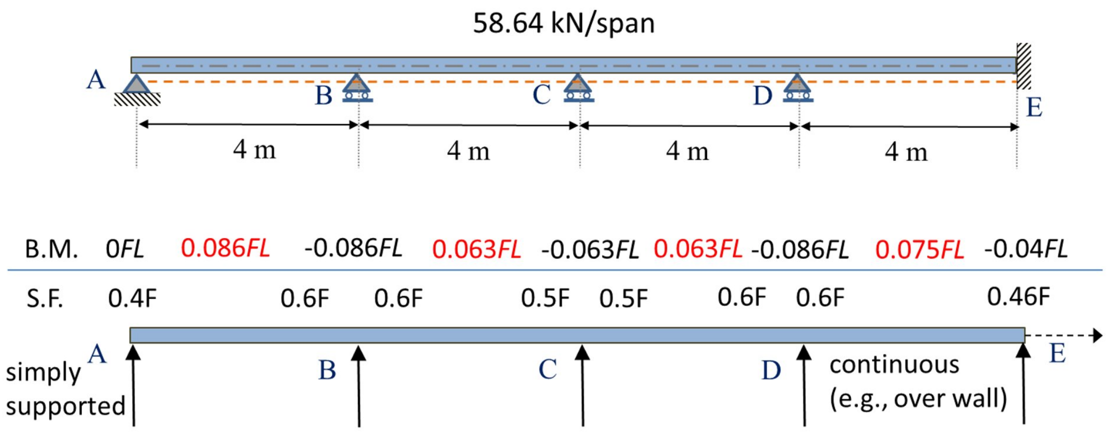

Figure 7.18: Application example of Table 7.2.

Application example of Table 7.2 Consider the continuous beam of Figure 7.18 as the structural system of (a typical 1 m wide strip of) an one-way slab. We use Table 7.2 to estimate the bending moment and shear force values along the structure for the design load combination.

- **Support A** is a hinged end support, or in other words, from the 'simple end support @outer support' column of Table 7.2:
  - moment expression: 0.0
  - shear expression: $0.4F = 0.4 \cdot 58.6 = 23.5\,\mathrm{kN}$
- **Midspan AB** is the midspan of an outer span, from the 'simple end support: near middle of end span' column of Table 7.2:
  - moment expression: $0.086FL = 0.086 \cdot 58.6 \cdot 4 = 20.2\,\mathrm{kN\,m}$
  - shear expression: 0.0
- **Support B** is a first interior support, from the '@1st interior support' column of Table 7.2:
  - moment expression: $-0.086FL = -0.086 \cdot 58.6 \cdot 4 = -20.2\,\mathrm{kN\,m}$
  - shear expression: $0.6F = 0.6 \cdot 58.6 = 35.2\,\mathrm{kN}$
- **Midspan BC** is the midspan of an interior span, from the 'middle interior spans' column of Table 7.2:
  - moment expression: $0.063FL = 0.063 \cdot 58.6 \cdot 4 = 14.8\,\mathrm{kN\,m}$

  - shear expression: 0.0

- **Support D** case is similar with support B.
- **Support C** is an interior support, from the 'interior support' column of Table 7.2:
  - moment expression: $-0.063FL = -0.063 \cdot 58.6 \cdot 4 = -14.8\,\mathrm{kN\,m}$
  - shear expression: $0.5F = 0.5 \cdot 58.6 = 29.3\,\mathrm{kN}$
- **Midspan DE** is the midspan of an outer span, from the 'continuous end support: near middle of end span' column of Table 7.2:
  - moment expression: $0.075FL = 0.075 \cdot 58.6 \cdot 4 = 17.6\,\mathrm{kN\,m}$
  - shear expression: 0.0
- **Support E** is a fixed end support, from the 'continuous end support: @outer support' column of Table 7.2:
  - moment expression: $-0.04FL = -0.04 \cdot 58.6 \cdot 4 = -9.4\,\mathrm{kN\,m}$
  - shear expression: $0.46F = 0.46 \cdot 58.6 = 27.3\,\mathrm{kN}$

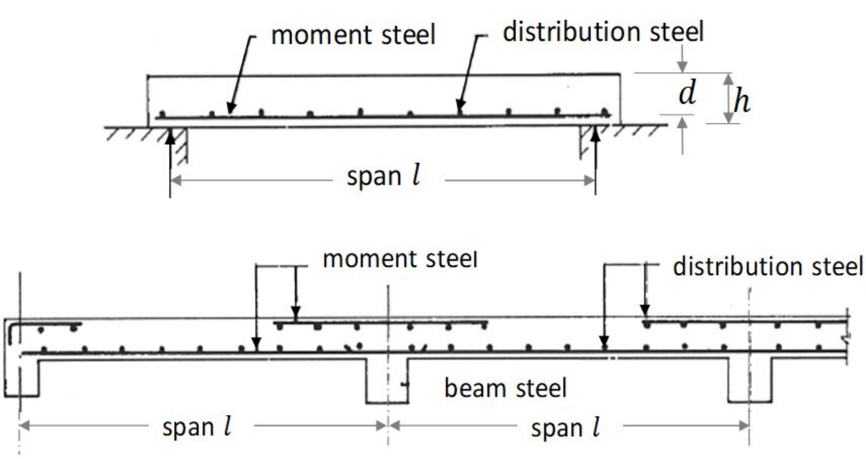

Figure 7.19: Moment and distribution steel in one-way slabs: simply supported (top) and continuous (bottom).

7.4.1 Moment Reinforcement and Distribution Reinforcement

Reinforcement in slabs is primarily categorized into moment steel, which resists bending moments, and distribution steel, which enhances overall structrual integrity and load distribution.

Moment Steel

Moment steel is the primary reinforcement designed to resist bending moments in slabs. Its placement and design are governed by the following practical guidelines:

- Moment-resisting reinforcement is provided only along the short span in one-way slabs, where bending is most significant.

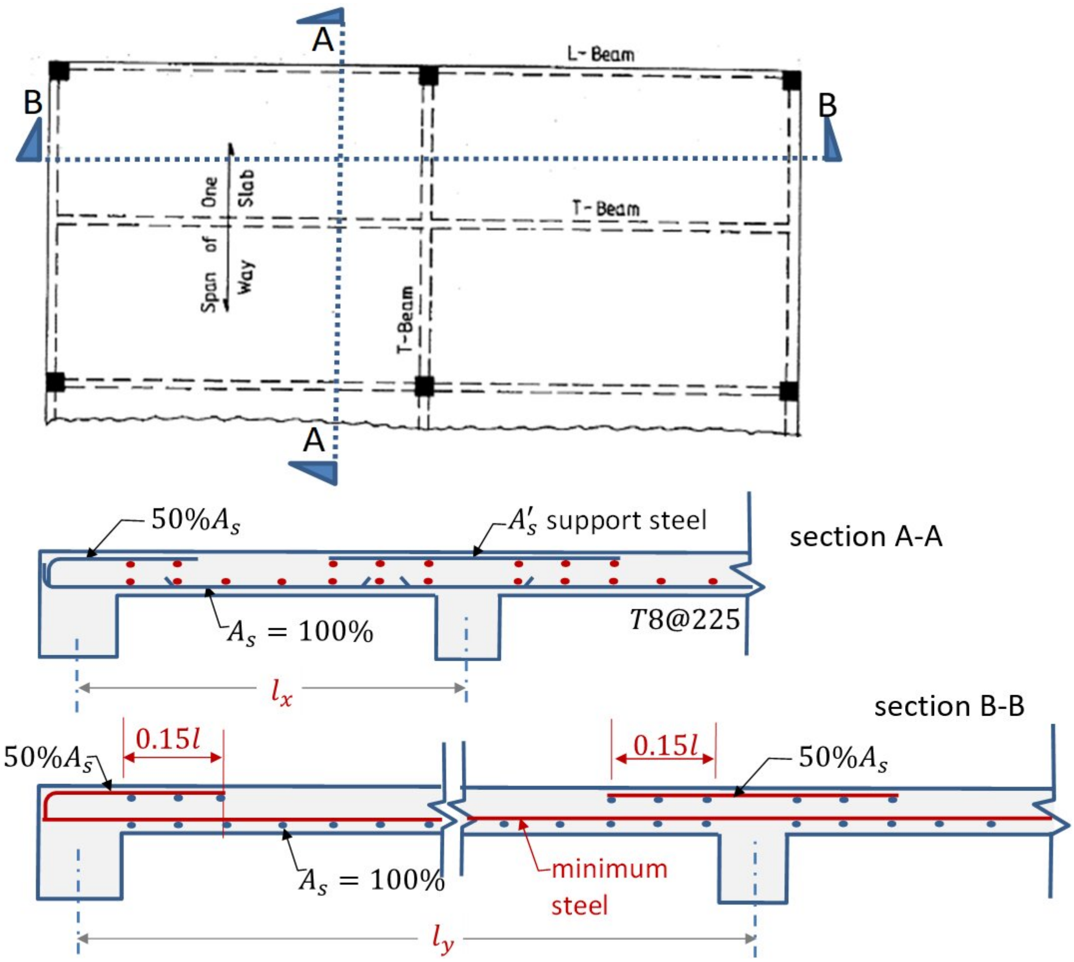

Figure 7.20: Moment and distribution steel in one-way slabs along two directions of a continuous slab.

- The minimum reinforcement ratio is $\rho_{\text{min}} = 0.13\%$ for high-yield steel or 0.24% for mild steel.
- Bar spacing is typically limited to $\leq \min\{250\,\text{mm}, 2h\}$, where h is the slab thickness, to ensure proper load distribution and crack control.

Distribution Steel

In one-way slabs, in addition to moment steel, distribution steel is provided perpendicular to the main reinforcement, as shown in Figs 7.19 and 7.20. This secondary reinforcement plays a critical role in maintaining the slab's performance under various loading conditions. The key purposes of distribution steel include:

- Tying the slab together to improve its monolithic behavior,
- Distributing non-uniform loads across the slab to prevent localized overstressing, and
- Accommodating potential bending moments along the longer span (which are not considered during design).

Distribution steel complements the moment steel by providing additional support in the transverse direction. Its design adheres to the following:

- Area $\geq$ minimum moment reinforcement ($\rho_{\text{min}} = 0.13\%$).
- Bar spacing is normally $\leq \min\{400\,\text{mm}, 3h\}$.

Shear Reinforcement

Shear reinforcement is typically unnecessary in slabs due to their geometry and loading conditions, but specific criteria must be considered:

- Under normal loading, shear stresses are rarely critical, so shear reinforcement is generally not required.
- For slabs less than $200\,\text{mm}$ thick, the shear stress v must not exceed the concrete's shear capacity v_c to avoid reinforcement.
- If $v \geq v_c$ in thicker slabs, shear reinforcement becomes necessary; however, it should not be used in slabs less than $200\,\text{mm}$ thick due to practical constraints.

Concrete Cover

Adequate concrete cover is essential to protect reinforcement from environmental factors and ensure durability. For grade 30 concrete under mild exposure conditions, the cover typically ranges from $20\,\text{mm}$ to $25\,\text{mm}$, balancing protection and structural efficiency.

Deflection and Crack Control

Controlling deflection and cracking is crucial for serviceability and long-term performance of slabs, achieved through simple yet effective design checks.

Deflection Control: Deflection is typically managed by verifying that the span-to-depth ratio remains within acceptable limits, a straightforward and reliable method for most slabs.

Crack Control: To limit crack widths and maintain aesthetics and durability, the following rules apply:

- The clear spacing of main steel should not exceed min$\{750\,\text{mm}, 3d\}$, where d is the effective depth.
- No additional checks are required under these common conditions:
  - For high-yield steel, when $d \leq 200\,\text{mm}$,
  - For mild steel, when $d \leq 250\,\text{mm}$,
  - When the reinforcement ratio $A_s/bd \geq 0.3\%$, ensuring sufficient steel to control cracking.

7.4.2 Example: Design of Continuous One-Way Slab

Design the one-way slab of Figure 7.21, assuming:

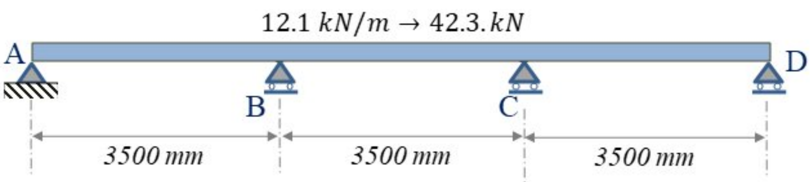

Figure 7.21: Continuous one-way slab

- 3 equal spans: $3.5\,\text{m}$ each.
- total slab depth is: $140\,\text{mm}$.
- Dead load = slab self-weight + screed + finish + ceiling + partitions = $5.2\,\text{kN/m}^2$.
- Imposed load = $3.0\,\text{kN/m}^2$.
- Materials: grade 30 concrete and grade 500 reinforcement.
- Mild condition of exposure $\rightarrow$ cover required = $25\,\text{mm}$.
- Design the slab and show the reinforcement on a sketch of the cross-section.

Solution

We calculate the required steel reinforcement for a typical cross-section of $1m$ width (see Figure 7.22). To calculate the **effective depth** of the cross-section assume:

- a reinforcement bar diameter of $\emptyset_L = 10\,\text{mm}$.
- then, the effective depth is $d = h - (c + \emptyset_L/2) = h - (25\,\text{mm} + 10\,\text{mm}/2) = 110\,\text{mm}$.

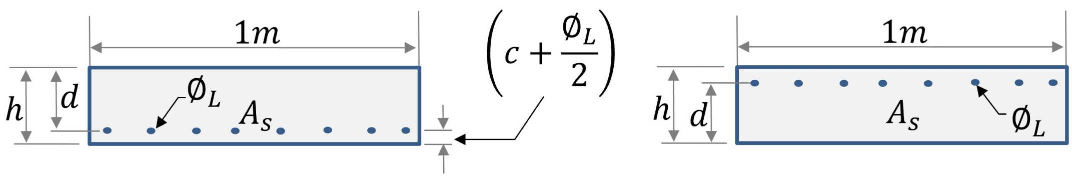

Figure 7.22: 1m wide typical slab cross-section for sagging (left) and hogging (right) bending moments

Design Load

- Design load per length on a 1.0 m wide strip = $(1.4 \cdot 5.2 + 1.6 \cdot 3) \cdot 1.0 = 12.1\,\mathrm{kN/m}$.
- Design load per span = $12.1 \cdot 3.5 = 42.3\,\mathrm{kN}$.
- We consider a single load case of maximum design load on all spans.

Design for flexure

Bending Moments and Shear Forces In lieu of structural analysis, the design bending moments and shear forces in the slab are calculated using Table 7.2 and summarized in Table 7.3.

supports/spans	bending moment $(\mathrm{kN\,m})$	shear force (kN)
A (outer support)	0	$0.40 \cdot 42.3 = 16.9$
AB (end span)	$0.086 \cdot 42.3 \cdot 3.5 = 12.7$	0
B (1st interior support)	$-0.086 \cdot 42.3 \cdot 3.5 = -12.7$	$0.60 \cdot 42.3 = 25.9$
BC (middle interior span)	$0.063 \cdot 42.3 \cdot 3.5 = 9.3$	0

Table 7.3: Design bending moments and shear forces for the examined structure.

Design of Moment Steel
Calculate the **minimum area** of reinforcement:

$$A_{s,\min} = 0.13\% \cdot 1000 \cdot 140 = 182\,\mathrm{mm^2/m}.$$

We need to account for moment redistribution of 20% $\rightarrow \beta_b = 0.8$, K' is (Eq 2.18):

$$K' = 0.405(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2 = 0.405 \cdot (0.8 - 0.4) - 0.18 \cdot (0.8 - 0.4)^2$$
$$K' = 0.133 \quad (\text{NOT} \quad K' = 0.156)$$

(a) **First interior support B**: $M_B = 12.7\,\mathrm{kN\,m/m}$ (Table 7.3) and $\beta_b = 0.8$:

- Design as a rectangular section of 1 m width.
- Calculate the dimensionless moment ratio:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{12.7 \cdot 10^6}{1000 \cdot 110^2 \cdot 30} = 0.035 < K' = 0.133$$

- Lever arm (Equation 2.8):

$$z = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) = 110\left(0.5 + \sqrt{0.25 - \frac{0.035}{0.9}}\right) = 109\,\mathrm{mm}$$

- Since $z > z_{\max} = 0.95d = 104\,\mathrm{mm}$ we take $z = z_{\max} = 104\,\mathrm{mm}$.
- Required steel (Equation 2.9):

$$A_s = \frac{12.7 \cdot 10^6}{0.87 \cdot 500 \cdot 104} = 281\,\mathrm{mm^2} \rightarrow 281\,\mathrm{mm^2/m} > A_{s,\min} = 182\,\mathrm{mm^2/m}$$

- Provide $8\,\mathrm{mm}$ bars at $150\,\mathrm{mm}$ centres (T8@150) $\rightarrow 335\,\mathrm{mm^2/m}$.

(b) **End span AB**: The design moment $M_{AB} = M_B = 12.7\,\mathrm{kN\,m/m}$ is the same as for the support B (Table 7.3) and the cross-section is symmetric to hogging and sagging moments (see Figure 7.22). Therefore, the calculations are identical with those of support B and do not need to be repeated. Provide the same reinforcement as at support B:
T8@150 ($A_s = 335\,\mathrm{mm^2/m}$).

(c) **Interior span BC**: $M_{BC} = 9.32\,\mathrm{kN\,m/m}$ (Table 7.3)

- As $M_{BC} < M_B \rightarrow z > 0.95d = 104\,\mathrm{mm}$, we take $z = z_{\max} = 104\,\mathrm{mm}$ and calculate the required area of reinforcement (Eq 2.9):

$$A_s = \frac{9.3 \cdot 10^6}{0.87 \cdot 500 \cdot 104} = 206\,\mathrm{mm^2/m} > A_{s,\min} = 182\,\mathrm{mm^2/m}$$

- Exceeds $A_{s,\min}$, so we provide T8@225 ($A_s = 223\,\mathrm{mm^2/m}$).

(d) **Outer support A**: According with the structural system of Figure 7.21 there is no moment on support A: $M_A = 0\,\mathrm{kN\,m/m}$. Hence in there should be no demand for moment steel at support A. However, steel is provided in the outer simple support of a one-way slab to:

- Resist small negative moments caused by the rotational (torsional) stiffness of the supporting beam or wall, which deviates from the idealized zero-moment assumption.
- Satisfy code-mandated minimum reinforcement for crack control and structural integrity.
- The top steel area of the outer support = $\frac{1}{2}$ the midspan steel = $281/2 = 141\,\mathrm{mm^2/m}$, but not less than the minimum area of $182\,\mathrm{mm^2/m}$ has to be provided → provide $8\,\mathrm{mm}$ bars at $250\,\mathrm{mm}$ centres (T8@250) to give $201\,\mathrm{mm^2/m}$.
- The tension bars in the bottom of outer support A are stopped off at the line of support.

The moment reinforcement follows, in general, the curtailment of steel bars of Figure 7.17.

Distribution Steel

- The minimum area of reinforcement ($A_{s,\min} = 182\,\mathrm{mm^2/m}$) has to be provided.
- The clear spacing between bars should not exceed $300\,\mathrm{mm}$ and $3d = 330\,\mathrm{mm}$, whichever is the lesser. Provide T8@250 to give $201\,\mathrm{mm^2/m}$, which is the same as that provided at the outer support A.

Shear Resistance (Usually Ignored in Practice)

Enhancement in design strength close to supports is not taken into account.

- **Outer support**: The shear resistance is based on the top (tension) steel T8@250 ($A_{s,\min} = 201\,\mathrm{mm^2/m}$):

$$\frac{100A_s}{b_v d} = \frac{100 \cdot 201}{1000 \cdot 110} = 0.183$$

The design concrete shear stress is (Equation 3.3):

$$v_c = \frac{0.79}{1.25}(0.183)^{1/3}\left(\frac{30}{25}\right)^{1/3}\left(\frac{400}{100}\right)^{1/4} = 0.53\,\mathrm{N/mm^2}$$

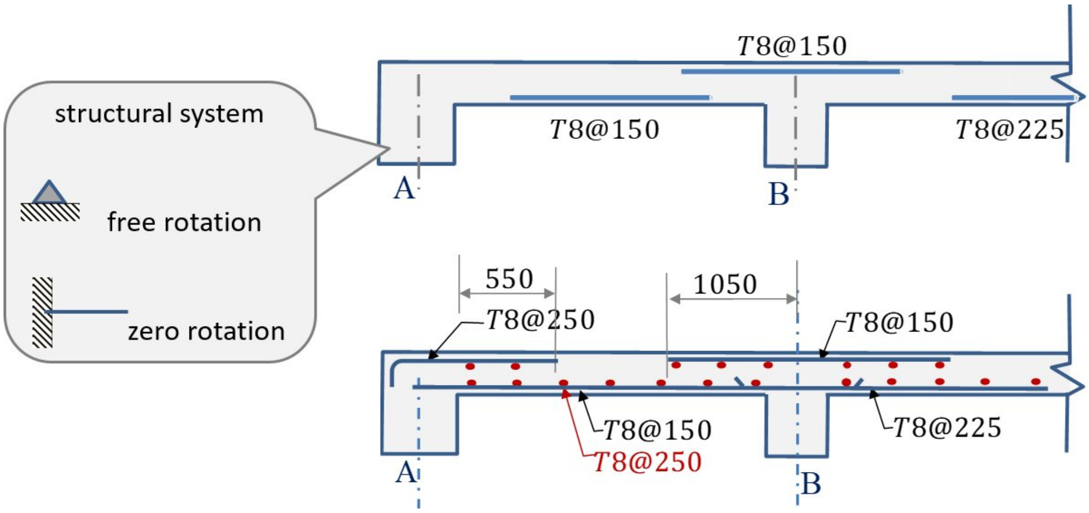

Figure 7.23: Top: moment steel in the slab. Bottom: Moment and redistribution steel in the slab

$$v = \frac{16.91 \cdot 10^3}{1000 \cdot 110} = 0.15 < 0.53\,\mathrm{N/mm^2}$$

$v < v_c$, no shear reinforcement is required.

- **Interior support**:

$$\frac{100A_s}{b_v d} = \frac{100 \cdot 281}{1000 \cdot 110} = 0.255; \quad v_c = 0.53\,\mathrm{N/mm^2}$$

$$v = \frac{25.37 \cdot 10^3}{1000 \cdot 110} = 0.23\mathrm{N/mm^2} < v_c$$

Therefore, no shear reinforcement is required.

SLS Deflection (Usually Ignored in Practice)

Enhancement in design strength close to supports is not taken into account.

- The span-depth ratio = $3500/140 = 25 <$ basic span/effective depth ratio $= 26$ for the continuous slab.
- The slab is therefore satisfactory with respect to deflection.

Or the slab is checked for deflection (see Notes: Optional course materials):

- The end span is checked here:
- The basic span/effective depth ratio is 26 for the continuous slab.

$$\frac{M}{bd^2} = 1.05; \quad f_s = \frac{5 \cdot 460 \cdot 304.4}{8 \cdot 314} = 278.7\,\mathrm{N/mm^2}$$

- The modification factor is:

$$0.55 + \frac{477 - 278.7}{120(0.9 + 1.05)} = 1.39$$

- Allowable span/depth ratio = $1.39 \cdot 26 = 36.1$.
- Actual span/depth ratio = $3500/110 = 31.8 < 36.1$, OK.

SLS Crack Control (Usually Ignored in Practice)

Because the steel grade is 500, the slab depth is less than $200\,\mathrm{mm}$, and the clear spacing does not exceed $3d = 330\,\mathrm{mm}$, the slab is satisfactory with respect to cracking.

7.5 Two-Way Solid Slabs

Two-way solid slabs are structural elements that span in two directions, distributing loads along both axes. Their behavior and reinforcement design depend on support conditions and span ratios. These slabs experience bending and torsional moments due to their double curvature.

Internal forces (bending and torsional moments, and shear forces) can be calculated using various methods:

- Analytical solutions based on elasticity theory for simple cases,
- Finite element analysis using specialized software,
- Czerny's tables,
- Markus method, or
- Simplified procedures in design codes.

This course emphasizes simplified procedures as an initial approach. While these methods provide a straightforward way to estimate internal forces, they are approximations with significant limitations. For educational purposes, we begin with simply supported two-way slabs before addressing the more practical case of restrained two-way slabs.

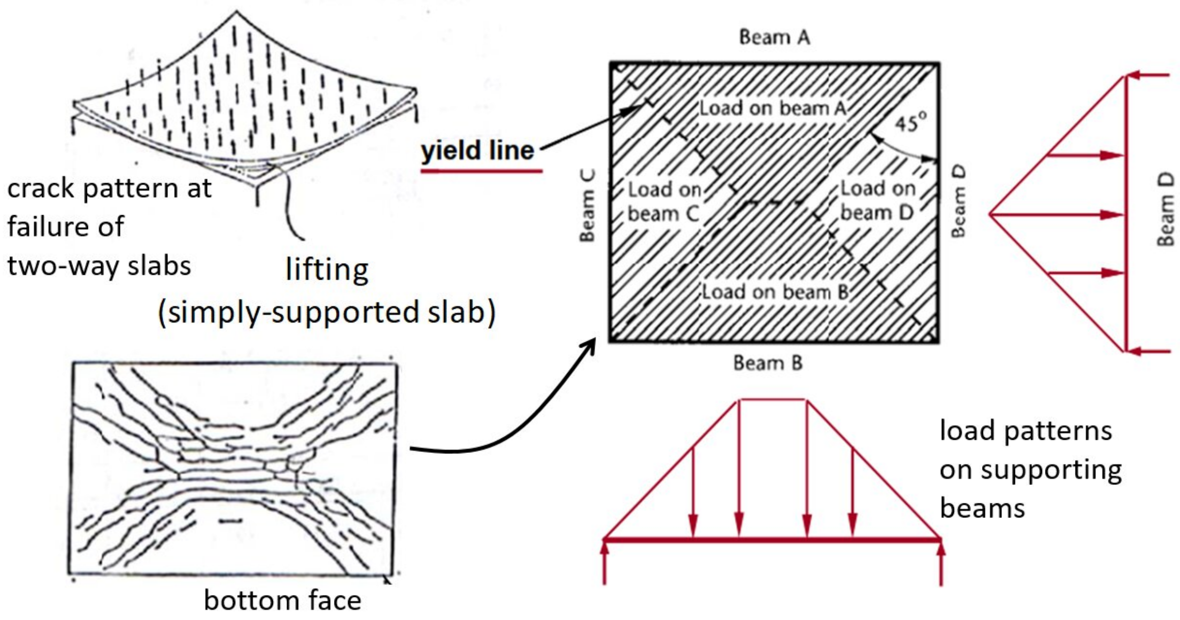

Figure 7.24: Two-way slabs: crack patterns at failure and load patterns on supporting beams

7.5.1 Markus Method

The Markus strip method offers a simplified approach for analyzing continuous two-way slabs by distributing the slab load p into components p_x and p_y along the x- and y-directions, respectively (see Figure 7.7). In the analysis of continuous strips using the Markus method, the loads p_x and p_y are applied to each span of a continuous strip, where the coefficients κ and ρ are calculated assuming fixed conditions at intermediate supports and simple end supports. However, the end supports can be assumed fixed if the cantilever length exceeds one-third of the slab length, i.e., $l_{\text{cantilever}} > \frac{1}{3} l_{\text{slab}}$.

These loads are defined as:

$$p_x = \kappa \cdot p, \quad p_y = \rho \cdot p,$$

where the coefficients κ and ρ (given in Table 7.4) ensure equal deflections ($w_x = w_y$) and are calculated as:

$$\kappa = \frac{\phi_y l_y^4}{\phi_x l_x^4 + \phi_y l_y^4}, \quad \rho = \frac{\phi_x l_x^4}{\phi_x l_x^4 + \phi_y l_y^4}.$$

The support coefficient ϕ depends on boundary conditions:

- $\phi = 5$: Simply supported,
- $\phi = 3$: One end fixed,
- $\phi = 1$: Both ends fixed.

The deflection w for a 1.0 m wide strip under load p with length l is:

- Simply supported: $w = \frac{5}{384} \frac{pl^4}{EI}$,
- One end fixed: $w = \frac{2}{384} \frac{pl^4}{EI}$,
- Both ends fixed: $w = \frac{1}{384} \frac{pl^4}{EI}$.

The loads p_x and p_y are applied to each span of a continuous strip, treated as a 1.0 m wide beam. The span moments, m_x and m_y in the x- and y-directions, determined from the analysis of a 1.0 m wide strip treated as a continuous beam, are reduced due to torsional moments m_{xy}. The reduction coefficients are defined as:

$$\nu_x = 1 - \frac{5}{6} \left(\frac{l_x}{l_y} \right)^2 \cdot \frac{m_x}{M_x}, \quad \nu_y = 1 - \frac{5}{6} \left(\frac{l_y}{l_x} \right)^2 \cdot \frac{m_y}{M_y},$$

where the reference moments M_x and M_y are:

$$M_x = \frac{p \cdot l_x^2}{8}, \quad M_y = \frac{p \cdot l_y^2}{8}.$$

This reduction is quantified by coefficients ν_x and ν_y, which depend on boundary conditions: $\nu \to 1$ along fixed edges (where torsional moments are negligible) and $\nu \to 0$ along simply supported edges (where torsional moments are significant).

Example: Reduction Coefficient (ν) Calculation

Consider a square two-way slab with $l_x = l_y$ (Figure 7.25), where the load distribution coefficients are $\kappa = 0.5$ and $\rho = 0.5$ (Table 7.4), so $p_x = \kappa \cdot p = 0.5 \cdot p$ and $p_y = \rho \cdot p = 0.5 \cdot p$. The reduction coefficient ν_x is calculated to adjust the bending moments due to torsional effects, with the following examples illustrating the impact of boundary conditions:

- For a simply supported strip:

$$m_x = \frac{p_x \cdot l_x^2}{8} = \frac{0.5 \cdot p \cdot l_x^2}{8} = \frac{p \cdot l_x^2}{16}, \quad M_x = \frac{p \cdot l_x^2}{8}.$$

The reduction coefficient is:

$$\nu_x = 1 - \frac{5}{6} \left(\frac{l_x}{l_y} \right)^2 \cdot \frac{m_x}{M_x} = 1 - \frac{5}{6} \cdot 1 \cdot \frac{\frac{p \cdot l_x^2}{16}}{\frac{p \cdot l_x^2}{8}} = 1 - \frac{5}{6} \cdot \frac{1}{2} = 0.583.$$

ℓ_y/ℓ_x	4 edges discontinuous		1 edge long continuous		2 edges long continuous	
	κ	ρ	κ	ρ	κ	ρ
0.50	0.0588	0.9412	0.0244	0.9756	0.0123	0.9877
0.52	0.0681	0.9319	0.0284	0.9716	0.0144	0.9856
0.54	0.0784	0.9216	0.0329	0.9671	0.0167	0.9833
0.56	0.0895	0.9105	0.0378	0.9622	0.0193	0.9807
0.58	0.1017	0.8983	0.0433	0.9567	0.0221	0.9779
0.60	0.1147	0.8853	0.0493	0.9507	0.0253	0.9747
0.62	0.1287	0.8713	0.0558	0.9442	0.0287	0.9713
0.64	0.1437	0.8563	0.0628	0.9371	0.0325	0.9675
0.66	0.1595	0.8405	0.0705	0.9295	0.0366	0.9634
0.68	0.1762	0.8238	0.0788	0.9212	0.0410	0.9590
0.70	0.1936	0.8064	0.0876	0.9124	0.0458	0.9542
0.72	0.2118	0.7882	0.0971	0.9029	0.0510	0.9490
0.74	0.2307	0.7693	0.1071	0.8929	0.0566	0.9434
0.76	0.2502	0.7498	0.1177	0.8823	0.0626	0.9374
0.78	0.2702	0.7298	0.1290	0.8710	0.0689	0.9311
0.80	0.2906	0.7094	0.1408	0.8592	0.0757	0.9243
0.82	0.3114	0.6886	0.1532	0.8468	0.0829	0.9171
0.84	0.3324	0.6676	0.1661	0.8339	0.0906	0.9094
0.86	0.3536	0.6464	0.1795	0.8205	0.0986	0.9014
0.88	0.3749	0.6251	0.1935	0.8065	0.1071	0.8929
0.90	0.3962	0.6038	0.2079	0.7921	0.1160	0.8840
0.92	0.4174	0.5826	0.2227	0.7773	0.1253	0.8747
0.94	0.4384	0.5616	0.2380	0.7620	0.1351	0.8649
0.96	0.4593	0.5407	0.2536	0.7464	0.1452	0.8548
0.98	0.4798	0.5202	0.2695	0.7305	0.1557	0.8443
1.00	0.5000	0.5000	0.2857	0.7143	0.1667	0.8333
1.05	0.5486	0.4514	0.3271	0.6729	0.1956	0.8044
1.10	0.5942	0.4058	0.3693	0.6307	0.2265	0.7735
1.15	0.6362	0.3638	0.4116	0.5884	0.2592	0.7408
1.20	0.6746	0.3254	0.4534	0.5466	0.2931	0.7069
1.25	0.7094	0.2906	0.4941	0.5059	0.3281	0.6719
1.30	0.7407	0.2593	0.5332	0.4668	0.3636	0.6364
1.35	0.7686	0.2314	0.5706	0.4294	0.3991	0.6009
1.40	0.7935	0.2065	0.6058	0.3942	0.4345	0.5655
1.45	0.8155	0.1845	0.6388	0.3612	0.4692	0.5308
1.50	0.8351	0.1649	0.6694	0.3306	0.5031	0.4969
1.55	0.8523	0.1477	0.6978	0.3022	0.5358	0.4642
1.60	0.8676	0.1324	0.7239	0.2761	0.5672	0.4328
1.65	0.8811	0.1189	0.7478	0.2522	0.5972	0.4028
1.70	0.8931	0.1069	0.7696	0.2304	0.6255	0.3745
1.75	0.9037	0.0963	0.7895	0.2105	0.6523	0.3477
1.80	0.9130	0.0870	0.8077	0.1923	0.6774	0.3226
1.85	0.9213	0.0787	0.8241	0.1759	0.7008	0.2992
1.90	0.9287	0.0713	0.8390	0.1610	0.7227	0.2773
1.95	0.9353	0.0647	0.8526	0.1474	0.7430	0.2570
2.00	0.9412	0.0588	0.8649	0.1351	0.7619	0.2381

Table 7.4: Table: coefficients of Markus method.

ℓ_y/ℓ_x	1 edge short continuous		2 edges adjacent continuous		1 edge short discontinuous	
	κ	ρ	κ	ρ	κ	ρ
0.50	0.1351	0.8649	0.0588	0.9412	0.0303	0.9697
0.52	0.1545	0.8455	0.0681	0.9319	0.0353	0.9647
0.54	0.1753	0.8247	0.0784	0.9216	0.0408	0.9592
0.56	0.1973	0.8027	0.0895	0.9105	0.0469	0.9531
0.58	0.2205	0.7795	0.1017	0.8983	0.0536	0.9464
0.60	0.2447	0.7553	0.1147	0.8853	0.0609	0.9391
0.62	0.2698	0.7302	0.1287	0.8713	0.0688	0.9312
0.64	0.2955	0.7045	0.1437	0.8563	0.0774	0.9226
0.66	0.3217	0.6783	0.1595	0.8405	0.0867	0.9133
0.68	0.3483	0.6517	0.1762	0.8238	0.0966	0.9034
0.70	0.3751	0.6249	0.1936	0.8064	0.1072	0.8928
0.72	0.4019	0.5981	0.2118	0.7882	0.1185	0.8815
0.74	0.4285	0.5715	0.2307	0.7693	0.1304	0.8696
0.76	0.4548	0.5452	0.2502	0.7498	0.1430	0.8570
0.78	0.4806	0.5194	0.2702	0.7298	0.1562	0.8438
0.80	0.5059	0.4941	0.2906	0.7094	0.1700	0.8300
0.82	0.5306	0.4694	0.3114	0.6886	0.1844	0.8156
0.84	0.5545	0.4455	0.3324	0.6676	0.1993	0.8007
0.86	0.5776	0.4224	0.3536	0.6464	0.2148	0.7852
0.88	0.5999	0.4001	0.3749	0.6251	0.2307	0.7693
0.90	0.6212	0.3788	0.3962	0.6038	0.2470	0.7530
0.92	0.6417	0.3583	0.4174	0.5826	0.2637	0.7363
0.94	0.6612	0.3388	0.4384	0.5616	0.2808	0.7192
0.96	0.6798	0.3202	0.4593	0.5407	0.2981	0.7019
0.98	0.6975	0.3025	0.4798	0.5202	0.3156	0.6844
1.00	0.7143	0.2857	0.5000	0.5000	0.3333	0.6667
1.05	0.7524	0.2476	0.5486	0.4514	0.3780	0.6220
1.10	0.7854	0.2146	0.5942	0.4058	0.4226	0.5774
1.15	0.8139	0.1861	0.6362	0.3638	0.4665	0.5335
1.20	0.8383	0.1617	0.6746	0.3254	0.5090	0.4910
1.25	0.8592	0.1408	0.7094	0.2906	0.5497	0.4503
1.30	0.8772	0.1228	0.7407	0.2593	0.5881	0.4119
1.35	0.8925	0.1075	0.7686	0.2314	0.6242	0.3578
1.40	0.9057	0.0943	0.7935	0.2065	0.6576	0.3424
1.45	0.9170	0.0830	0.8155	0.1845	0.6885	0.3115
1.50	0.9268	0.0732	0.8351	0.1649	0.7168	0.2832
1.55	0.9352	0.0648	0.8523	0.1477	0.7427	0.2573
1.60	0.9425	0.0575	0.8676	0.1324	0.7662	0.2338
1.65	0.9488	0.0512	0.8811	0.1189	0.7875	0.2125
1.70	0.9543	0.0457	0.8931	0.1069	0.8068	0.1932
1.75	0.9591	0.0409	0.9037	0.0963	0.8242	0.1758
1.80	0.9631	0.0367	0.9130	0.0870	0.8400	0.1600
1.85	0.9670	0.0330	0.9213	0.0787	0.8542	0.1458
1.90	0.9702	0.0298	0.9287	0.0713	0.8670	0.1330
1.95	0.9731	0.0269	0.9353	0.0647	0.8785	0.1215
2.00	0.9756	0.0244	0.9412	0.0588	0.8889	0.1111

Table 7.4 continued: coefficients of Markus method.

ℓ_y/ℓ_x	2 edges short continuous		1 edge long discontinuous		4 edges continuous	
	κ	ρ	κ	ρ	κ	ρ
0.50	0.2381	0.7619	0.1111	0.8889	0.0588	0.9412
0.52	0.2677	0.7323	0.1276	0.8724	0.0681	0.9319
0.54	0.2983	0.7017	0.1453	0.8547	0.0784	0.9216
0.56	0.3296	0.6704	0.1644	0.8356	0.0895	0.9105
0.58	0.3614	0.6386	0.1846	0.8154	0.1017	0.8983
0.60	0.3932	0.6068	0.2058	0.7942	0.1147	0.8853
0.62	0.4249	0.5751	0.2281	0.7719	0.1287	0.8713
0.64	0.4562	0.5438	0.2512	0.7488	0.1437	0.8563
0.66	0.4868	0.5132	0.2751	0.7249	0.1595	0.8405
0.68	0.5167	0.4833	0.2995	0.7005	0.1762	0.8238
0.70	0.5456	0.4544	0.3244	0.6756	0.1936	0.8064
0.72	0.5733	0.4267	0.3496	0.6504	0.2118	0.7882
0.74	0.5999	0.4001	0.3749	0.6251	0.2307	0.7693
0.76	0.6252	0.3748	0.4002	0.5998	0.2502	0.7498
0.78	0.6492	0.3508	0.4254	0.5746	0.2702	0.7298
0.80	0.6719	0.3281	0.4503	0.5497	0.2906	0.7094
0.82	0.6933	0.3067	0.4749	0.5251	0.3114	0.6886
0.84	0.7134	0.2866	0.4989	0.5011	0.3324	0.6676
0.86	0.7323	0.2677	0.5224	0.4776	0.3536	0.6464
0.88	0.7499	0.2501	0.5453	0.4547	0.3749	0.6251
0.90	0.7664	0.2336	0.5675	0.4325	0.3962	0.6038
0.92	0.7818	0.2182	0.5889	0.4111	0.4174	0.5826
0.94	0.7961	0.2039	0.6096	0.3904	0.4384	0.5616
0.96	0.8094	0.1906	0.6295	0.3705	0.4593	0.5407
0.98	0.8218	0.1782	0.6485	0.3515	0.4798	0.5202
1.00	0.8333	0.1667	0.6667	0.3333	0.5000	0.5000
1.05	0.8587	0.1413	0.7085	0.2915	0.5486	0.4514
1.10	0.8798	0.1202	0.7454	0.2546	0.5942	0.4058
1.15	0.8974	0.1026	0.7777	0.2223	0.6362	0.3638
1.20	0.9120	0.0880	0.8057	0.1943	0.6746	0.3254
1.25	0.9243	0.0757	0.8300	0.1700	0.7094	0.2906
1.30	0.9346	0.0652	0.8510	0.1490	0.7407	0.2593
1.35	0.9432	0.0564	0.8692	0.1308	0.7686	0.2314
1.40	0.9505	0.0492	0.8848	0.1152	0.7935	0.2065
1.45	0.9567	0.0433	0.8984	0.1016	0.8155	0.1845
1.50	0.9620	0.0380	0.9101	0.0899	0.8351	0.1649
1.55	0.9665	0.0332	0.9203	0.0797	0.8523	0.1477
1.60	0.9704	0.0296	0.9291	0.0709	0.8676	0.1324
1.65	0.9737	0.0263	0.9368	0.0632	0.8811	0.1189
1.70	0.9766	0.0234	0.9435	0.0565	0.8931	0.1069
1.75	0.9791	0.0209	0.9494	0.0506	0.9037	0.0963
1.80	0.9813	0.0187	0.9545	0.0455	0.9130	0.0870
1.85	0.9832	0.0168	0.9591	0.0409	0.9213	0.0787
1.90	0.9849	0.0151	0.9631	0.0369	0.9287	0.0713
1.95	0.9864	0.0136	0.9666	0.0334	0.9353	0.0647
2.00	0.9877	0.0123	0.9697	0.0303	0.9412	0.0588

Table 7.4 continued: coefficients of Markus method.

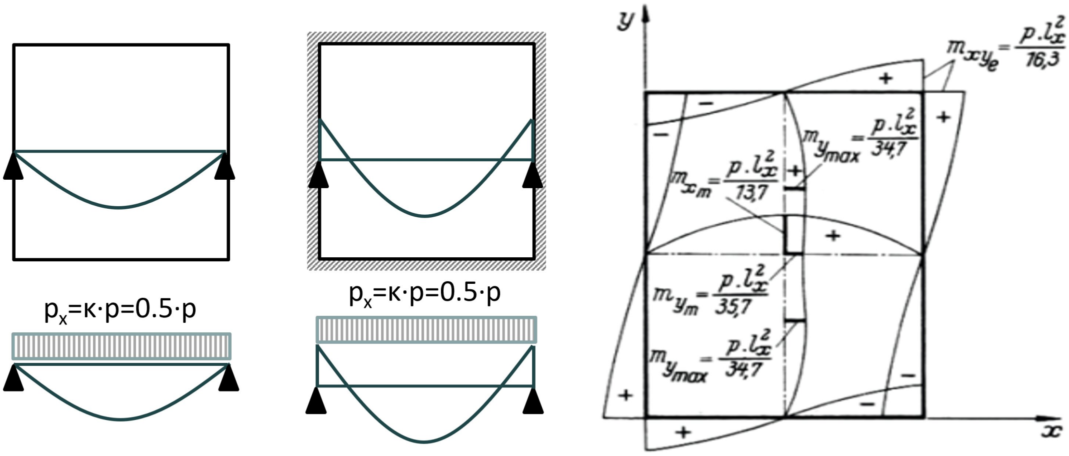

Figure 7.25: Square slab $l_x = l_y$ with simple supports (left), fixed supports (middle), and a sample torsional moment diagram (right).

- For a strip with both ends fixed:

$$m_x = \frac{p_x \cdot l_x^2}{24} = \frac{0.5 \cdot p \cdot l_x^2}{24} = \frac{p \cdot l_x^2}{48}, \quad M_x = \frac{p \cdot l_x^2}{8}.$$

  Thus:

$$\nu_x = 1 - \frac{5}{6} \cdot 1 \cdot \frac{\frac{p \cdot l_x^2}{48}}{\frac{p \cdot l_x^2}{8}} = 1 - \frac{5}{6} \cdot \frac{1}{6} = 0.861.$$

These results illustrate that torsional moments are negligible along fixed edges ($\nu \to 1$) and significant along simply supported edges ($\nu \to 0$).

7.5.2 Simply Supported Slabs

To start from the simplest case, consider a slab spanning in two directions at right angles, simply supported on all four edges. When the spans are of similar length $l_x \approx l_y$, double curvature develops along the entire side, causing the slab to deflect about both axes (Figure 7.26 left). This deflection leads to the corners lifting and curling upward from the supports, inducing torsional moments. Consequently, bending moments develop in both directions, and torsional moments arise due to restraints at the strip edges. As a result, similar reinforcement is required in both directions to resist these effects.

For simply supported slabs where no provision is made to prevent corner lifting or resist torsion, the span moments are estimated as follows. The maximum moments per unit width (1 m) at mid-span are (Figure 7.26 right). :

$$m_{sx} = \alpha_{sx} n l_x^2, \quad \alpha_{sx} = \frac{(l_y/l_x)^4}{8[1 + (l_y/l_x)^4]}, \quad A_{sx} = \frac{m_{sx}}{0.87 f_y z},$$
$$m_{sy} = \alpha_{sy} n l_x^2, \quad \alpha_{sy} = \frac{(l_y/l_x)^2}{8[1 + (l_y/l_x)^4]}, \quad A_{sy} = \frac{m_{sy}}{0.87 f_y z},$$

where:

- l_x and l_y are the shorter and longer spans, respectively,
- $n = 1.4G_k + 1.6Q_k$ is the design load,

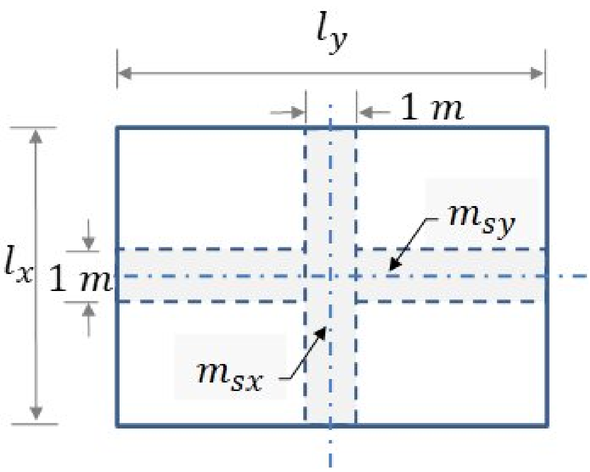

Figure 7.26: Centre strips of simply-supported two-way slab

- A_{sx} and A_{sy} are the required reinforcement areas per meter width, or design charts may be used to determine them.

These coefficients are derived from bending moment coefficients for slabs spanning in two directions at right angles, simply supported on four sides, as provided in Table 7.5 ([HKCC2013: Table 6.5]). In the **edge strips** of such slabs, only nominal reinforcement ($A_{s,\min}$) is necessary, while the **middle strips** require detailed reinforcement.

l_y/l_x	1.0	1.1	1.2	1.3	1.4	1.5	1.75	2.0
α_{sx}	0.062	0.074	0.084	0.093	0.099	0.104	0.113	0.118
α_{sy}	0.062	0.061	0.059	0.055	0.051	0.046	0.037	0.029

Table 7.5: Table of values for l_y/l_x, α_{sx}, and α_{sy}.

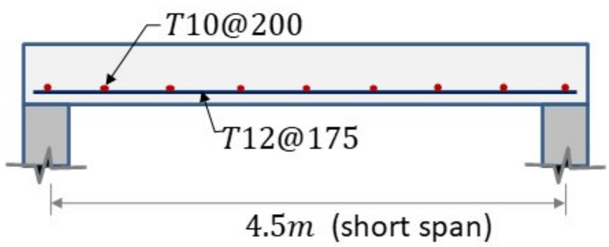

Figure 7.27: Simply supported slab: always arrange short-span steel bars at the outer layer to utilize the the larger effective depth.

Illustrative Reinforcement Detailing:

Always arrange short-span steel bars (l_x) at the outer layer to optimize load resistance (Figure 7.27).

7.5.3 Restrained Slabs

In restrained slabs, each direction is divided into middle and edge strips. The maximum design moments, m_{sx} and m_{sy}, are applied only to the middle strips and are calculated as:

$$m_{sx} = \beta_{sx} n l_x^2, \quad m_{sy} = \beta_{sy} n l_x^2, \tag{7.1}$$

where the moment coefficients β_{sx} and β_{sy} are provided in Table 7.6 ([HKCC2013: Table 6.6]), derived from a modified yield-line analysis for rectangular panels supported on four sides with torsion provision at corners.

In restrained slabs a few reinforcement rules are:

- The steel bars in the **middle strips** are uniformly spaced.
- At discontinuous edges, top steel equal to half the bottom steel at mid-span is provided for crack control.
- In the **edge strips**, only nominal reinforcement is required ($\rho_{\text{min}} = 0.13\%$ for high-yield steel, 0.24% for mild steel), supplemented by torsion reinforcement as specified below.

7.5.4 Example: Two-Way Restrained Solid Slab

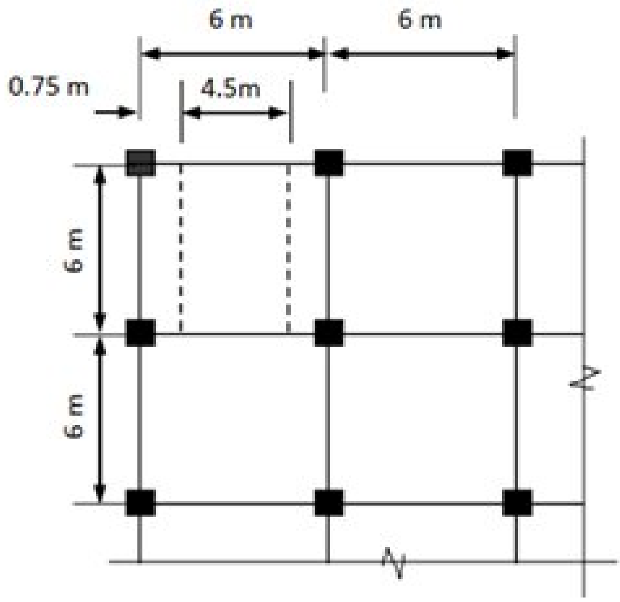

Figure 7.28: Part of floor plan (Bhatt et al (2006): 8.6.6 Example)

Consider the floor plan with restrained slabs and edge beams of Figure 7.28. The slab is $180\,\text{mm}$ thick, with a total dead load of $6.2\,\text{kN/m}^2$ and an imposed load of $2.5\,\text{kN/m}^2$. Material properties are $f_{cu} = 30\,\text{N/mm}^2$ and $f_y = 500\,\text{N/mm}^2$. The task is to design the corner slab and illustrate reinforcement on sketches.

Slab Division, Moments, and Reinforcement:

The corner slab is divided into middle and edge strips. The design load is:

$$n = 1.4G_k + 1.6Q_k = 1.4 \cdot 6.2 + 1.6 \cdot 2.5 = 12.7\,\text{kN/m}^2.$$

Assuming $10\,\text{mm}$ diameter bars and $25\,\text{mm}$ concrete cover:

Table 7.6: [HKCC2013: Table 6.6] Bending moment coefficients for rectangular slabs (restrained) supported on 4 sides.

Types of panel	Moments	Short span coefficient β_{sx} for l_y/l_x								Long span β_{sy}
		1.0	1.1	1.2	1.3	1.4	1.5	1.75	2.0	
4 edges continuous	negM @ cont edge	0.031	0.037	0.042	0.046	0.050	0.053	0.059	0.063	0.032
	posM @ midspan	0.024	0.028	0.032	0.035	0.037	0.040	0.044	0.048	0.024
1 short edge discontinuous	negM @ cont edge	0.039	0.044	0.048	0.052	0.055	0.058	0.063	0.067	0.037
	posM @ midspan	0.029	0.033	0.036	0.039	0.041	0.043	0.047	0.050	0.028
1 long edge discontinuous	negM @ cont edge	0.039	0.049	0.056	0.062	0.068	0.073	0.082	0.089	0.037
	posM @ midspan	0.030	0.036	0.042	0.047	0.051	0.055	0.062	0.067	0.028
2 adjacent edges discontinuous	negM @ cont edge	0.047	0.056	0.063	0.069	0.074	0.078	0.087	0.093	0.045
	posM @ midspan	0.036	0.042	0.047	0.051	0.055	0.059	0.065	0.070	0.034
2 short edges discontinuous	negM @ cont edge	0.046	0.050	0.054	0.057	0.060	0.062	0.067	0.070	–
	posM @ midspan	0.034	0.038	0.040	0.043	0.045	0.047	0.050	0.053	0.034
2 long edges discontinuous	negM @ cont edge	–	–	–	–	–	–	–	–	0.045
	posM @ midspan	0.034	0.046	0.056	0.065	0.072	0.078	0.091	0.100	0.034
3 edges discnt 1 long edge cont	negM @ cont edge	0.057	0.065	0.071	0.076	0.081	0.084	0.092	0.098	–
	posM @ midspan	0.043	0.048	0.053	0.057	0.060	0.063	0.069	0.074	0.044
3 edges discnt 1 short edge cont	negM @ cont edge	–	–	–	–	–	–	–	–	0.058
	posM @ midspan	0.042	0.054	0.063	0.071	0.078	0.084	0.096	0.105	0.044
4 edges discnt	negM @ cont edge	–	–	–	–	–	–	–	–	–
	posM @ midspan	0.055	0.065	0.074	0.081	0.087	0.092	0.103	0.111	0.056

negM @ cont edge = negative moment at continuous edge;
posM @ midspan = positive moment at midspan

- Outer layer: $d = 180 - 30 = 150\,\mathrm{mm}$,
- Inner layer: $d = 150 - 10 = 140\,\mathrm{mm}$,
- Minimum tension steel area: $A_{s,\min} = \frac{0.13 \cdot 1000 \cdot 180}{100} = 234\,\mathrm{mm}^2/\mathrm{m}$.

Since the slab is square (symmetric), only one direction needs analysis for middle strip moments and steel areas. The moment coefficients from Table 7.6 ([HKCC2013: Table 6.6])

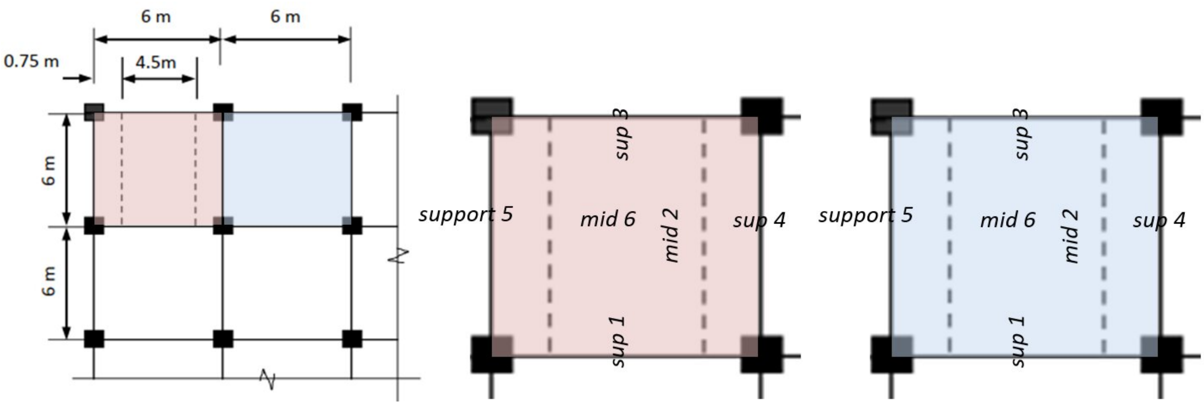

Figure 7.29: Moment coefficients

for the top left and the top middle slab of the floor plan of Figure 7.29:

	top left slab	top middle slab
support 1:	$\beta_{SX} = -0.047$	$\beta_{SX} = -0.039$
support 3:	$\beta_{SX} = 0$	$\beta_{SX} = 0$
support 4:	$\beta_{SX} = -0.047$	$\beta_{SX} = -0.039$
support 5:	$\beta_{SX} = 0$	$\beta_{SX} = -0.039$
midspan 2:	$\beta_{SX} = 0.036$	$\beta_{SX} = 0.029$
midspan 6:	$\beta_{SX} = 0.036$	$\beta_{SX} = 0.029$

Table 7.7: Coefficients β_{SX} for the top left and the top middle slap of Figure 7.29.

Moment Steel Reinforcement:

Unlike one-way slabs, two-way slabs require moment steel along both spans. At the midspan, reinforcement bars are placed at the bottom in both directions, necessitating a decision regarding which direction will form the exterior bottom layer and which will form the interior bottom layer. When the spans differ in length, the reinforcement for the shorter span should be positioned as the exterior (bottom) layer, and that of the long-span as the interior (bottom) layer. The exterior layer benefits from a slightly larger effective depth compared to the interior layer, as illustrated in Figure 7.31 (bottom right). In this specific example the slab is square so the choice of exterior versus interior layer direction is arbitrary. Nevertheless, once we choose a direction to place as exterior or interior bottom layer we should be consistent in our calculations.

In the following we design the reinforcement of specifically the top left slab of Figure 7.29. Similar to the design of one-way slabs, the design calculations for two-way slabs utilize a typical rectangular cross-section with a width of 1 m (as in Figure 7.22), yielding the required steel reinforcement per meter.

(a) **Supports 1 and 4 (Continuous Edges):** $d = 150\,\mathrm{mm}$, $l_x = 6\,\mathrm{m}$,

$$m_{sx} = \beta_{sx} n l_x^2 = -0.047 \cdot 12.7 \cdot 6^2 = -21.5\,\text{kNm/m},$$

$$K = \frac{M}{bd^2 f_{cu}} = \frac{21.5 \cdot 10^6}{30 \cdot 1000 \cdot 150^2} = 0.032 < K' = 0.156,$$

$$\frac{z}{d} = 0.5 + \sqrt{0.25 - \frac{0.032}{0.9}} = 0.96 > 0.95 = z_{max} \rightarrow z = z_{max} = 0.95 \cdot 150 = 143\,\mathrm{mm},$$

$$A_s = \frac{M}{0.87 f_y z} = \frac{21.5 \cdot 10^6}{0.87 \cdot 500 \cdot 143} = 342\,\mathrm{mm}^2/\mathrm{m}.$$

Provide T10@200 ($A_s = 393\,\mathrm{mm}^2/\mathrm{m}$) top steel.

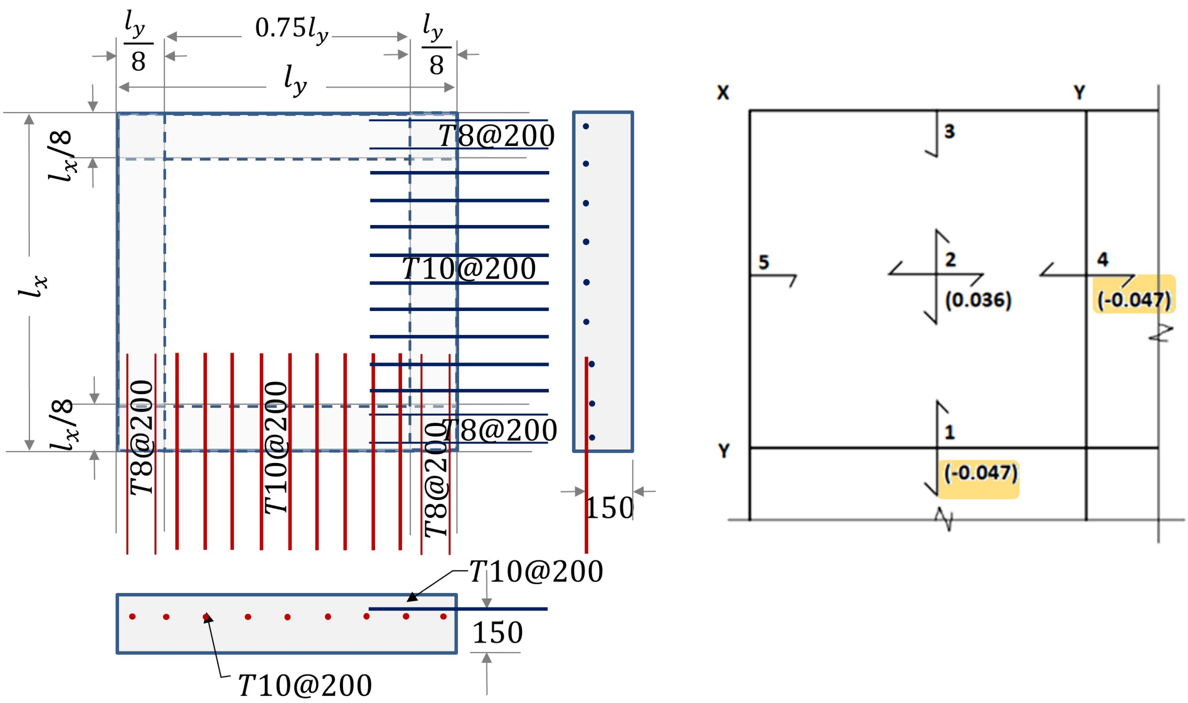

Figure 7.30: Slab supports 1 and 4 reinforcement: bottom steel T10@150

(b) **Span 2:** interior bottom layer $d = 140\,\mathrm{mm}$,

$$m_{sx} = \beta_{sx} n l_x^2 = 0.036 \cdot 12.7 \cdot 6^2 = 16.5\,\text{kNm/m},$$

$$K = \frac{M}{bd^2 f_{cu}} = \frac{16.5 \cdot 10^6}{30 \cdot 1000 \cdot 140^2} = 0.028 < 0.156,$$

$$\frac{z}{d} = 0.5 + \sqrt{0.25 - \frac{0.028}{0.9}} = 0.97 > 0.95 = z_{max} \rightarrow z = z_{max} = 0.95 \cdot 140 = 133\,\mathrm{mm},$$

$$A_s = \frac{M}{0.87 f_y z} = \frac{16.5 \cdot 10^6}{0.87 \cdot 500 \cdot 133} = 285\,\mathrm{mm}^2/\mathrm{m}.$$

Provide bottom steel T8@150 ($A_s = 335\,\mathrm{mm}^2/\mathrm{m}$) or
T8@400 + T10@400 ($A_s = 322\,\mathrm{mm}^2/\mathrm{m}$).

(c) **Supports 3 and 5 (Discontinuous Edges):** $d = 150\,\mathrm{mm}$,

$$A_s = 0.5 \cdot 285 < A_{s,\min} = 234\,\mathrm{mm}^2/\mathrm{m}.$$

Provide T8@200 ($A_s = 251\,\mathrm{mm}^2/\mathrm{m}$) top steel.

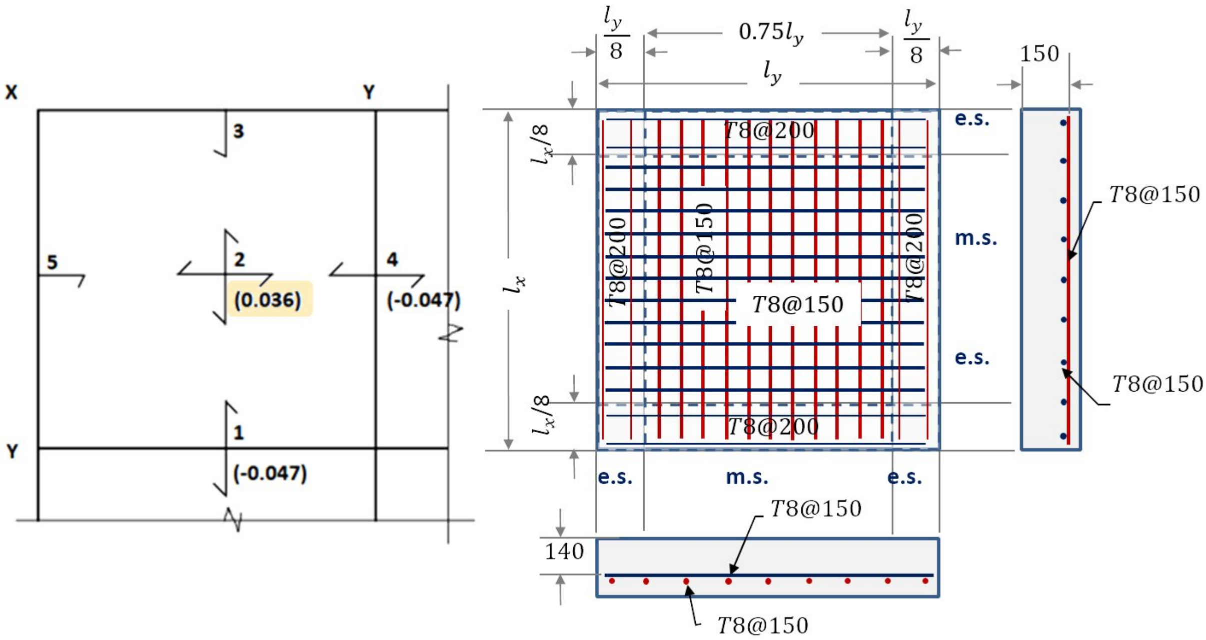

Figure 7.31: Slab midspan reinforcement: bottom steel T8@150

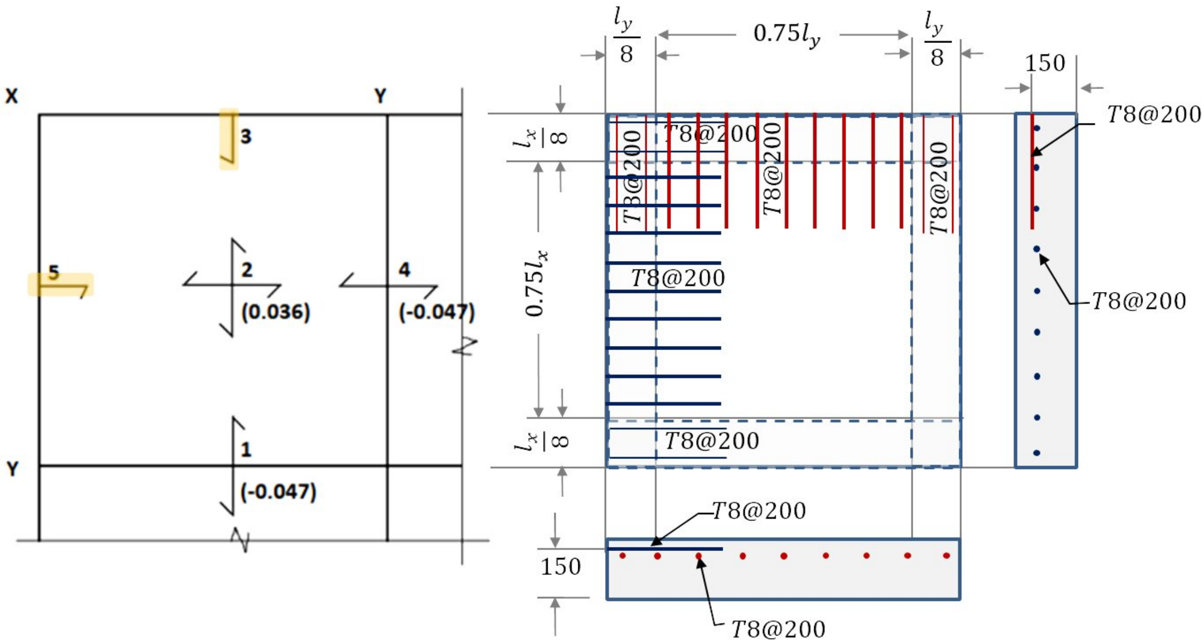

Figure 7.32: Slab supports 3 and 5 reinforcement: top steel T8@200

(d) In detailing, moment steel is not curtailed since both negative and positive steel would fall below $A_{s,\min}$ if 50% of bars were cut off.

Shear Forces and Shear Resistance

The design loads on supporting beams are:

$$v_{sx} = \beta_{vx} n l_x, \quad v_{sy} = \beta_{vy} n l_x,$$

where shear coefficients β_{vx} and β_{vy} are from Table 7.1 ([HKCC2013: Table 6.7]).

For the square top left slab of Figure 7.29 shear coefficients are:

- Support 1: $\beta_{vx} = 0.40$,
- Support 3: $\beta_{vx} = 0.26$,
- Support 4: $\beta_{vx} = 0.40$,
- Support 5: $\beta_{vx} = 0.26$.

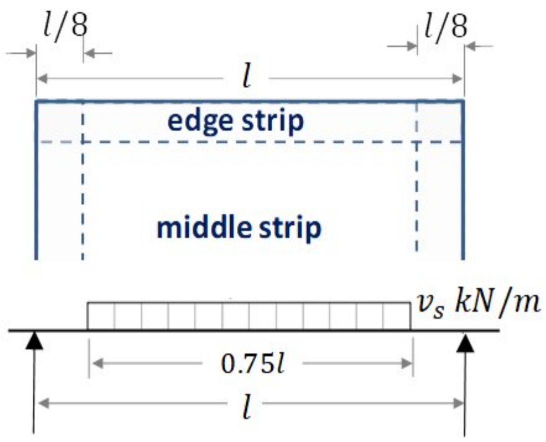

Figure 7.33: 0

(a) **Supports 1 and 4:**

- load:$v_{sx} = \beta_{vx} \cdot n \cdot l_x = 0.4 \cdot 12.7 \cdot 6 = 30.5$ kN/m,
- shear stress:

$$v = \frac{v_{sx}}{b \cdot d} = \frac{30.5 \cdot 10^3}{1000 \cdot 150} = 0.20\,\mathrm{N/mm^2},$$

- design concrete shear stress, with tension (top) steel T10@200 ($A_s = 393\,\mathrm{mm^2/m}$) $\frac{100A_s}{b \cdot d} = \frac{100 \cdot 393}{1000 \cdot 150} = 0.26$: $v_c = 0.57\,\mathrm{N/mm^2}$.
- Since $v < v_c$, no shear reinforcement is required.

(b) **Supports 3 and 5:**

- load:$v_{sx} = \beta_{vx} \cdot n \cdot l_x = 0.26 \cdot 12.7 \cdot 6 = 19.8$ kN/m,
- shear stress:

$$v = \frac{v_{sx}}{b \cdot d} = \frac{19.8 \cdot 10^3}{1000 \cdot 150} = 0.13\,\mathrm{N/mm^2},$$

- design concrete shear stress, $v_c = 0.49\,\mathrm{N/mm^2}$.
- Since $v < v_c$, no shear reinforcement is required.

Torsion Reinforcement:

At discontinuous corners where the slab is simply supported on both edges (e.g., Corner X) and lifting is prevented, torsion reinforcement is required. For Corner Y (discontinuous on one side):

- Corner X: $A_{s,X} = \frac{3}{4}A_{sx} = \frac{3}{4} \cdot 285 = 214\,\mathrm{mm^2/m}$,
- Corner Y: $A_{s,Y} = \frac{3}{8}A_{sx} = \frac{3}{8} \cdot 285 = 107\,\mathrm{mm^2/m}$.

Provide T8@200 ($A_s = 251\,\mathrm{mm^2/m}$) in top and bottom meshes, extending $\frac{1}{5}l_x = 1.2\,\mathrm{m}$ from edges. Edge strips also receive T8@200 as minimum reinforcement.

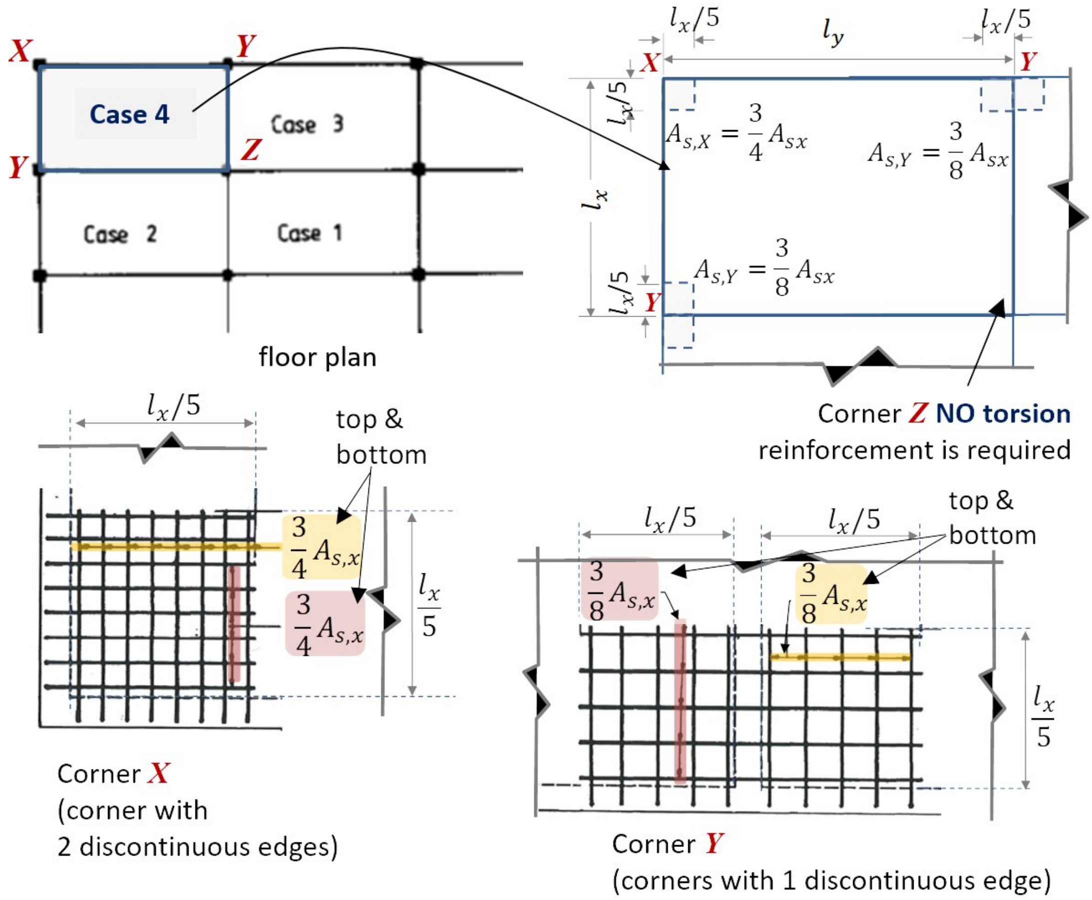

Figure 7.34: top: Corners X and Y require torsion reinforcement. Corner Z does not.
bottom: Required torsion reinforcement at corners X and Y.

SLS Deflection (Usually Ignored):

From [HKCC2013: Table 7.4], with $d = 150\,\mathrm{mm}$. Check using steel at mid-span with:

- $d = 150\,\mathrm{mm}$
- Basic $span/depth$ ratio = 26
- $\frac{m_{sx}}{bd^2} = \frac{16.5 \cdot 10^6}{1000 \cdot 150^2} = 0.73$

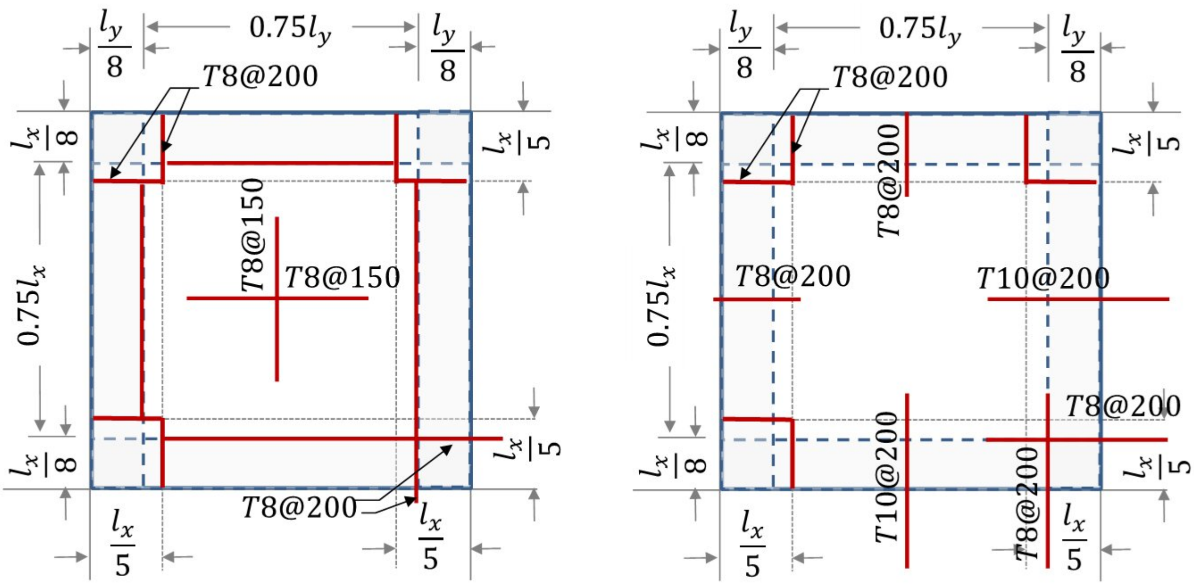

Figure 7.35: Left: required bottom reinforcement, right: required top reinforcement

- The design service stress in the tension reinforcement in a member may be estimated from the equation:

$$f_s = \frac{2}{3} f_y \frac{A_{st,req}}{A_{st,prov}} = \frac{2}{3} \cdot 500 \cdot \frac{285}{322} = 295\,\mathrm{N/mm^2}$$

- The modification factor = $0.55 + \frac{477 - 295}{120(0.9 + 0.73)} = 1.48$
- Allowable $span/d$ ratio = $1.48 \cdot 26 \approx 38.48$
- Actual $span/d$ ratio = $\frac{6000}{150} = 40$

Consequently, the slab can be considered to be almost satisfactory.

SLS Cracking (Usually Ignored):

Bar spacing $\leq 3d = 3 \cdot 140 = 420\,\mathrm{mm}$, and for grade 500 steel with $d < 200\,\mathrm{mm}$, no further checks are required.

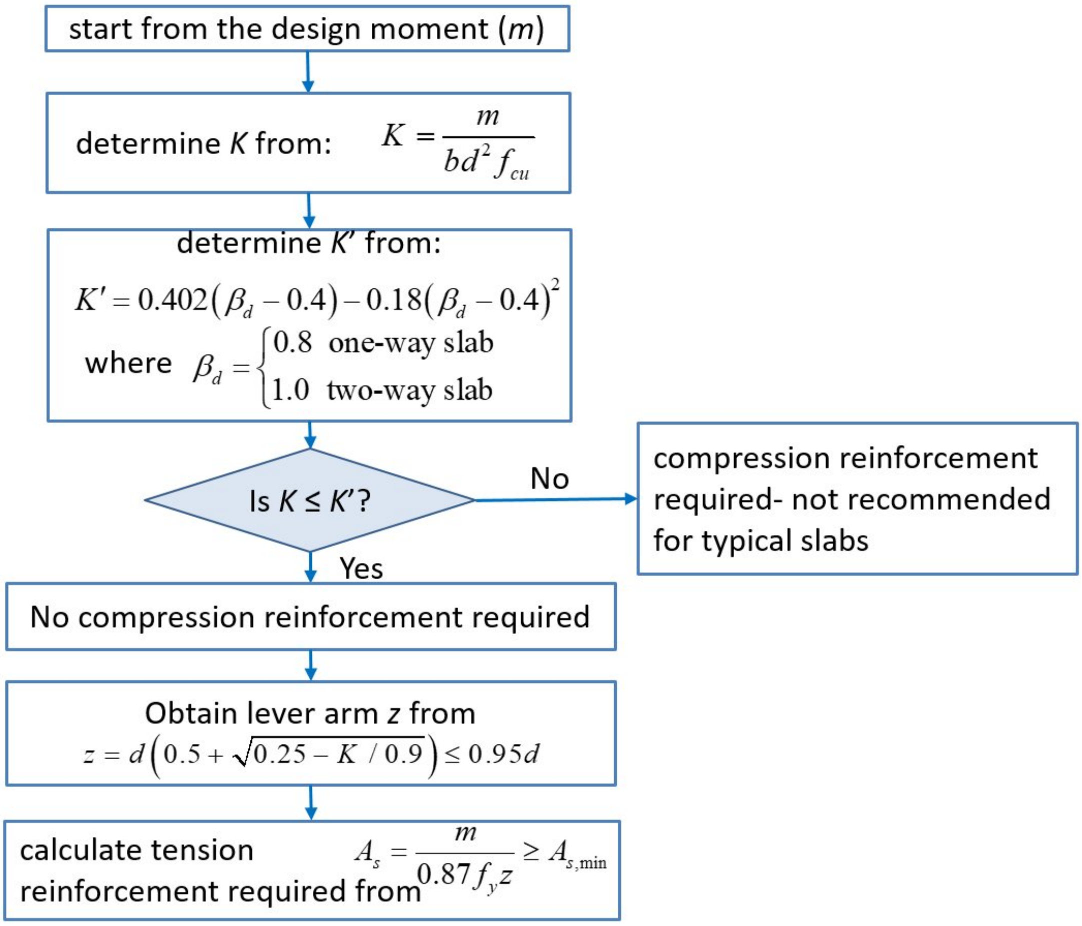

Figure 7.36: Design moment steel flowchart for slabs.

7.6 Punching Shear Failure

So far, we have only dealt with uniformly applied area loads on slabs. A concentrated load P on a slab induces shearing stresses around a section near the load, a phenomenon known as punching shear. Punching shear failure can occur in flat-slab constructions, column footings, pile caps, and bridge decks (Figure 7.37). It is a brittle and catastrophic type of failure.

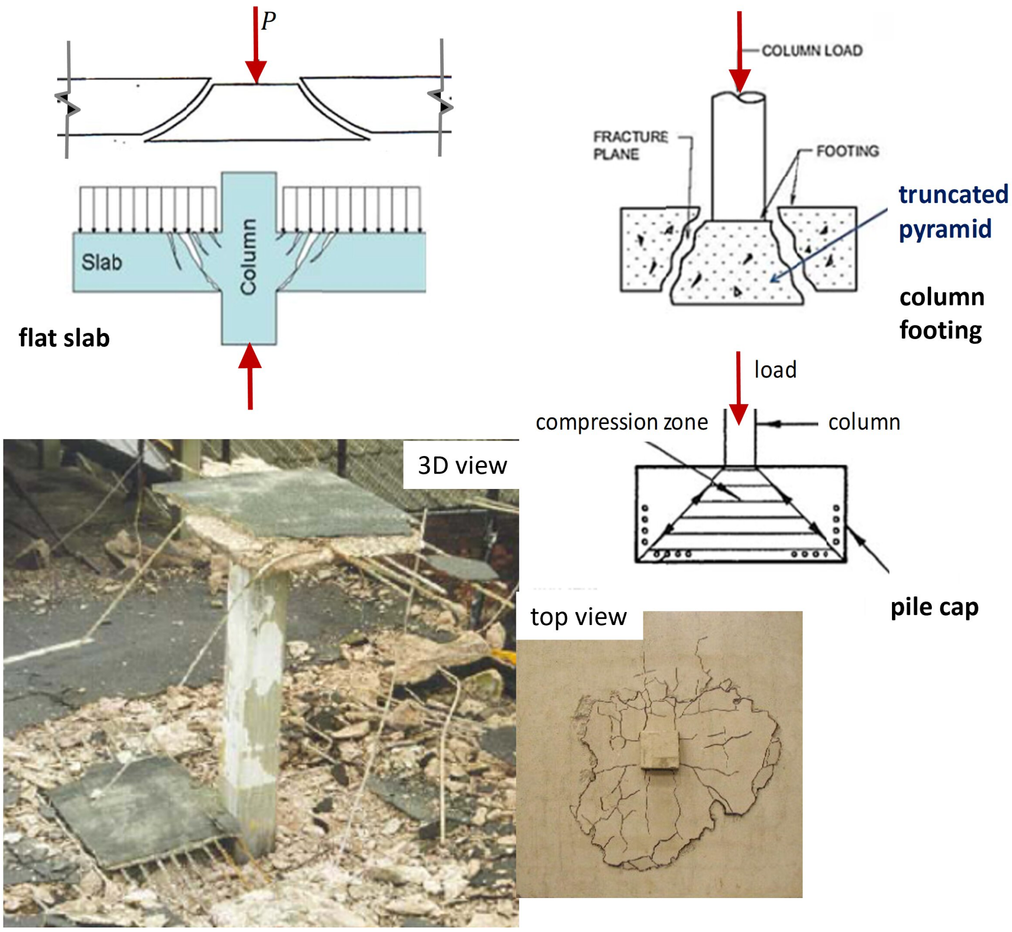

Figure 7.37: Common cases of punching shear in reinforced concrete structures: flat slabs, column footing, pile caps

Analysis

To design against punching shear failure we have to ensure sufficient resistance. The control perimeter for punching shear is defined at a distance of $1.5d$ from the loaded area, with square corners regardless of whether the loaded area is square or circular (Figure 7.39). The punching shear stress is calculated as:

$$v = \frac{P}{ud}$$

where u is the **control perimeter** (i.e., the perimeter of the critical section), given by:

$$u = 4 \cdot (2 \cdot 1.5d + c) \tag{7.2}$$

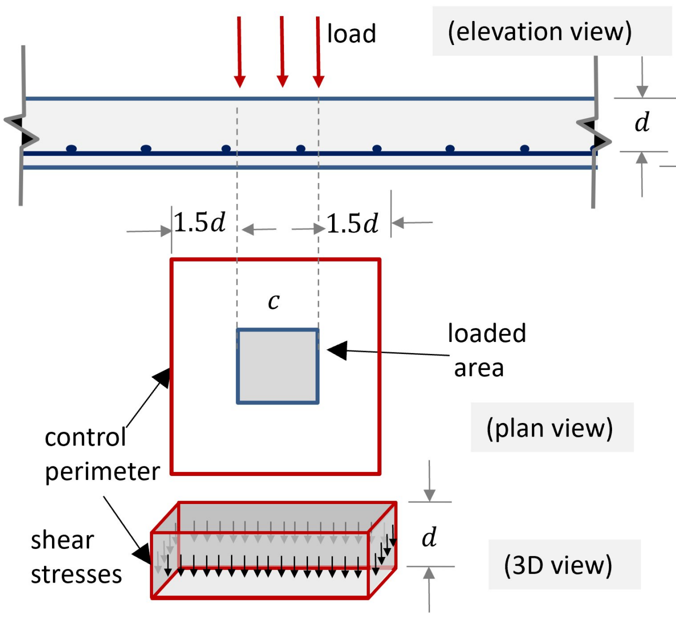

Figure 7.38: Slab subjected to punching shear and shear stresses on the control perimeter/surface

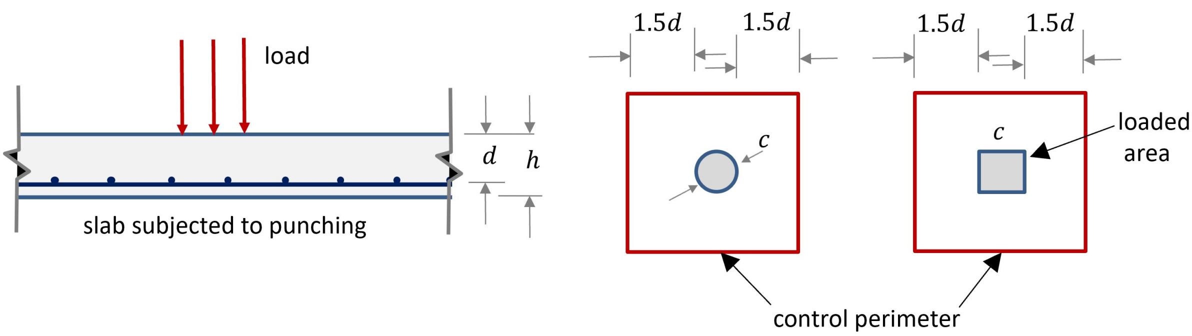

Figure 7.39: Punching shear control perimeter with different loaded areas

If the punching shear stress $v < v_c$, no shear reinforcement is required. Additionally, checks must ensure that the shear stress at the perimeter of the loaded area's face does not exceed:

$$v \leq v_{\max} = \min\{0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}\} \tag{7.3}$$

7.6.1 Example 1: Calculation of Ultimate Punching Load

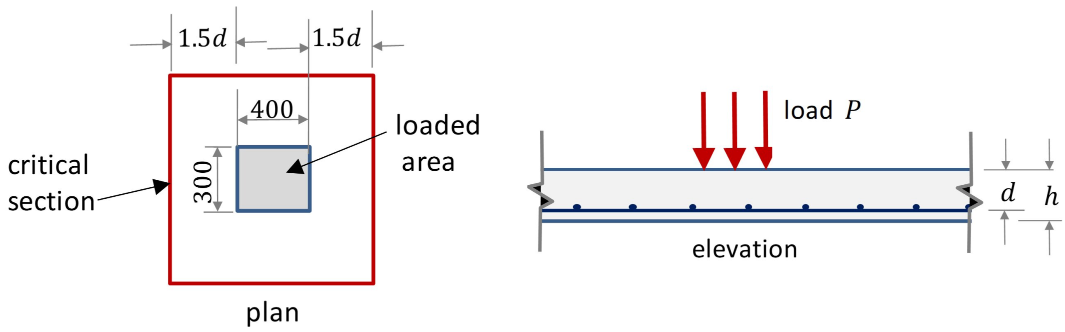

Figure 7.40: Example problem

Consider a reinforced concrete (RC) slab with the following properties:

- Thickness: $180\,\mathrm{mm}$,
- Effective depth: $d = 150\,\mathrm{mm}$,
- Grade 30 concrete ($f_{cu} = 30\,\mathrm{N/mm^2}$),
- Reinforcement: T12@150 one way ($A_s = 754\,\mathrm{mm^2}$) and T10@200 the other way ($A_s = 393\,\mathrm{mm^2}$),
- Loaded area: $300\,\mathrm{mm} \times 400\,\mathrm{mm}$.

We aim to calculate the punching-load capacity without exceeding the ultimate shear stress.

Solution

To calculate the resistance of the slab to punching shear we first need to calculate the design concrete shear strength. To this end, we need to calculate the effect of flexural reinforcement through reinforcement ratio, for a typical 1m width. Since we have different reinforcement in the two horizontal directions, we calculate both ratios and proceed with the average value (ad hoc assumption):

- For T12@150: $\dfrac{100A_s}{bd} = \dfrac{100 \cdot 754}{1000 \cdot 150} = 0.50$,
- For T10@200: $\dfrac{100A_s}{bd} = \dfrac{100 \cdot 393}{1000 \cdot 150} = 0.26$.

The average steel ratio is: $\dfrac{0.50 + 0.26}{2} = 0.38$. We will proceed the using this average.

- the design concrete shear strength v_c is ($\gamma_m = 1.25$):

$$v_c = \frac{0.79}{\gamma_m}\left(\frac{f_{cu}}{25}\right)^{1/3}\left(\frac{100A_s}{b_v d}\right)^{1/3}\left(\frac{400}{d}\right)^{1/4} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{1/3}(0.38)^{1/3}\left(\frac{400}{150}\right)^{1/4} =$$

$$v_c = 0.62\,\mathrm{N/mm^2}$$

- The **control punching shear perimeter** is:

$$u = 2 \cdot [(400 + 3d) + (300 + 3d)] = 2 \cdot [(400 + 450) + (300 + 450)] = 3200\,\mathrm{mm}$$

- The maximum punching load is:

$$P = v_c \cdot u \cdot d = 0.62 \cdot 3200 \cdot 150 \cdot 10^{-3} = 297.6\,\mathrm{kN}$$

- Check the shear stress at the face of the loaded area ($u = 2 \cdot (300 + 400) = 1400\,\mathrm{mm}$):

$$v = \frac{P}{u \cdot d} = \frac{297.6 \cdot 10^3}{1400 \cdot 150} = 1.42\,\mathrm{N/mm^2} < 0.8\sqrt{f_{cu}} = 0.8\sqrt{30} = 4.38\,\mathrm{N/mm^2}$$

  which is well below the maximum allowable shear stress of $4.38\,\mathrm{N/mm^2}$, confirming the design is safe.

7.6.2 Example 2: Calculation of Slab Thickness Required

Consider an RC slab with:

- Grade 30 concrete ($f_{cu} = 30\,\mathrm{N/mm^2}$),
- Reinforcement: T12@150 in both directions ($A_s = 754\,\mathrm{mm^2}$),
- Punching load: $330\,\mathrm{kN}$,
- Loaded area: $300 \times 300\,\mathrm{mm}$.

Determine the required slab thickness against punching shear failure.

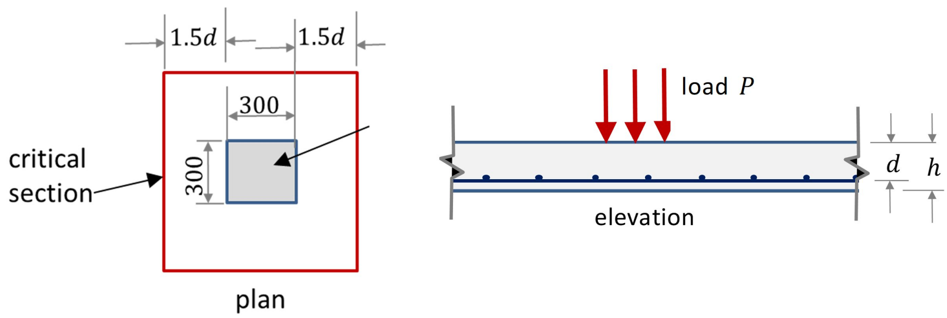

Figure 7.41: Example problem

Solution

To calculate the punching-load capacity we need to calculate the design concrete shear strength. To this end, we need the depth of the slab which we do not have. Instead, we will assume an initial effective depth $d = 150\,\mathrm{mm}$ to calculate v_c and check it in retrospect.

- The reinforcement ratio is

$$\frac{100A_s}{bd} = \frac{100 \cdot 754}{1000 \cdot 150} = 0.50$$

- The design concrete shear strength is

$$v_c = \frac{0.79}{\gamma_m}\left(\frac{f_{cu}}{25}\right)^{1/3}\left(\frac{100A_s}{b_v d}\right)^{1/3}\left(\frac{400}{d}\right)^{1/4} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{1/3}(0.50)^{1/3}\left(\frac{400}{150}\right)^{1/4} \rightarrow$$

$$v_c = 0.68\,\mathrm{N/mm^2}$$

- The punching shear perimeter is:

$$u = 4 \cdot (300 + 2 \cdot 1.5d) = 4 \cdot (300 + 3d) = (1200 + 12d)\,\mathrm{mm}$$

- Solving the equation for the applied load for the effective depth d of the slab:

$$330 \cdot 10^3 = 0.68 \cdot (1200 + 12d) \cdot d \rightarrow 12d^2 + 1200d - 485294 = 0 \rightarrow$$

  We arrive at the quadratic formula for the required effective depth d:

$$d = \frac{-1200 \pm \sqrt{1200^2 + 4 \cdot 12 \cdot 485294}}{2 \cdot 12} \approx 157\,\mathrm{mm}$$

- and from that the required total depth h of the slab:

$$h = d + \text{cover} + \text{bar radius} = 157 + 35 = 192\,\mathrm{mm}$$

- Take $h = 200\,\mathrm{mm}$, so $d = 200 - 35 = 165\,\mathrm{mm}$.
- Check the shear stress at the face ($u = 4 \cdot 300 = 1200\,\mathrm{mm}$):

$$v = \frac{330 \cdot 10^3}{1200 \cdot 165} = 1.67\,\mathrm{N/mm^2} < 0.8\sqrt{30} = 4.38\,\mathrm{N/mm^2}$$

  This is satisfactory.

Discussion

In practice, steel ratios in slabs typically range from 0.25% to 0.5%. When designing slab thickness for punching loads, a preliminary steel ratio may be assumed if unknown. For instance, if the steel ratio were 0.35%:

$$v_c = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{1/3}(0.35)^{1/3}\left(\frac{400}{150}\right)^{1/4} = 0.60\,\mathrm{N/mm^2}$$

Recalculating with $330 \cdot 10^3 = 0.60 \cdot (1200 + 12d) \cdot d$, $d \approx 160\,\mathrm{mm}$, leading to $h = 200\,\mathrm{mm}$, which aligns with the previous result.

7.6.3 Punching Shear Design for Shear Reinforcement

[Clauses 6.1.5.7(e) and (f)]

When $v_c < v < 2v_c$, shear reinforcement in the form of links may be provided to increase the shear resistance of the punching shear resistance, where:

- The slab thickness should be over $200\,\mathrm{mm}$ (IStructE recommends the slab thickness be $\geq 250\,\mathrm{mm}$).
- The first perimeter (Figure 7.42) is checked:
  - If $v \leq v_c$, no further checks are required.
  - If $v > v_c$, successive perimeters have to be checked until one is reached where $v \leq v_c$.

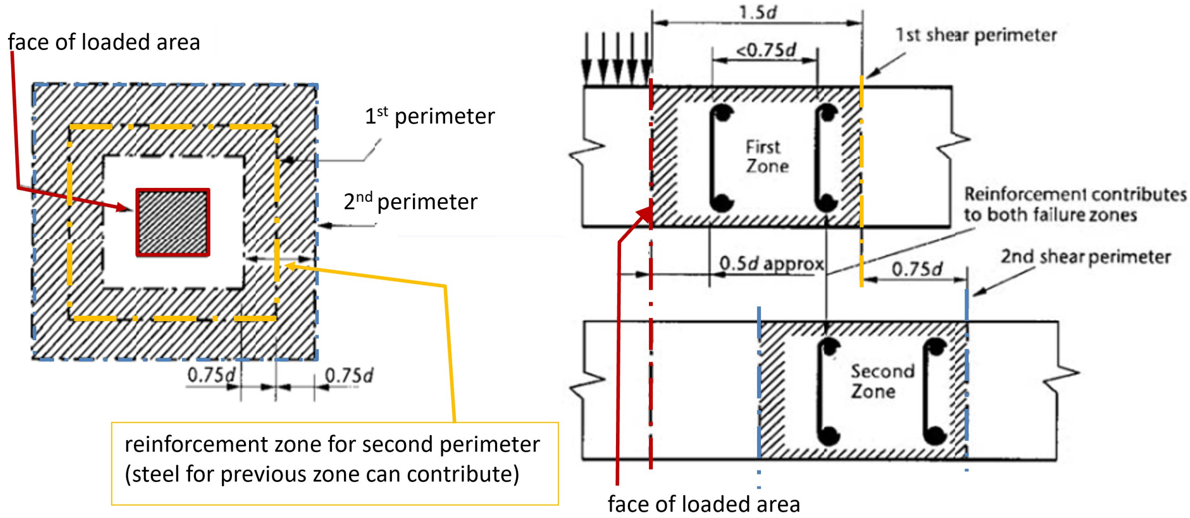

Figure 7.42: Punching shear reinforcement zones with the corresponding perimeters.

Punching Shear Reinforcement

Punching shear reinforcement in punching shear zones should be provided on at least two perimeters (Figure 7.44) between the column perimeter and the first perimeter, where:

- The **1st perimeter of reinforcement** should:
  - Be located approximately $0.5d$ from the face of the column.
  - Contain not less than 40% of A_{sv}.
- The **2nd perimeter of reinforcement** should be located $\leq 0.75d$ from the 1st perimeter.
- Reinforcement zone for the second perimeter (steel from the previous zone can contribute).
- The spacing of perimeters of reinforcement should be $\leq 0.75d$.
- The reinforcement is distributed evenly around a perimeter, and the spacing of the legs of links should be $\leq 1.5d$.

Amount of Punching Shear Reinforcement:

- For cases where $v \leq 1.6v_c$:

$$\sum A_{sv} \geq \frac{(v - v_c)ud}{0.87f_y}$$

- For cases where $1.6v_c < v \leq 2v_c$:

$$\sum A_{sv} \geq \frac{5(0.7v - v_c)ud}{0.87f_y}$$

- Minimum amount:

$$\left(\sum A_{sv}\right)_{\text{min}} \geq \frac{0.4ud}{0.87f_y}$$

  where:

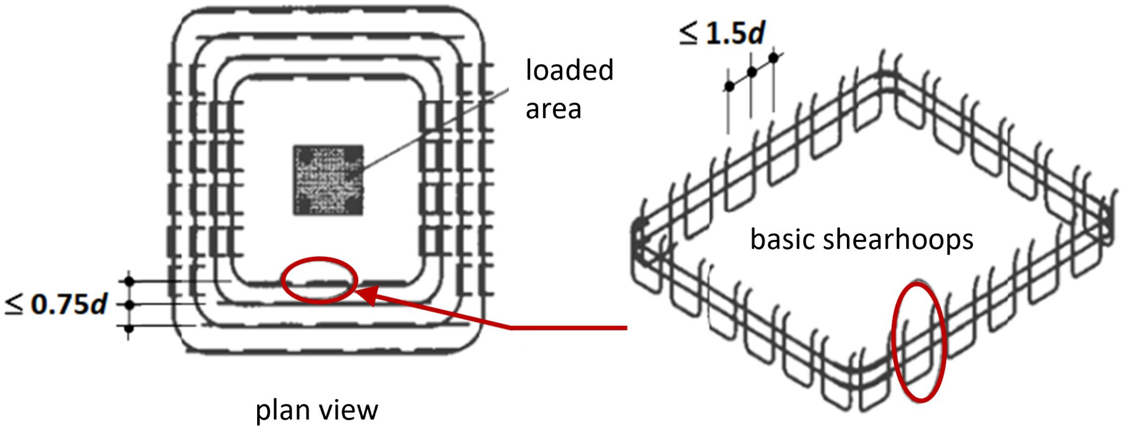

Figure 7.43: Shearhoop Reinforcement Detailing.

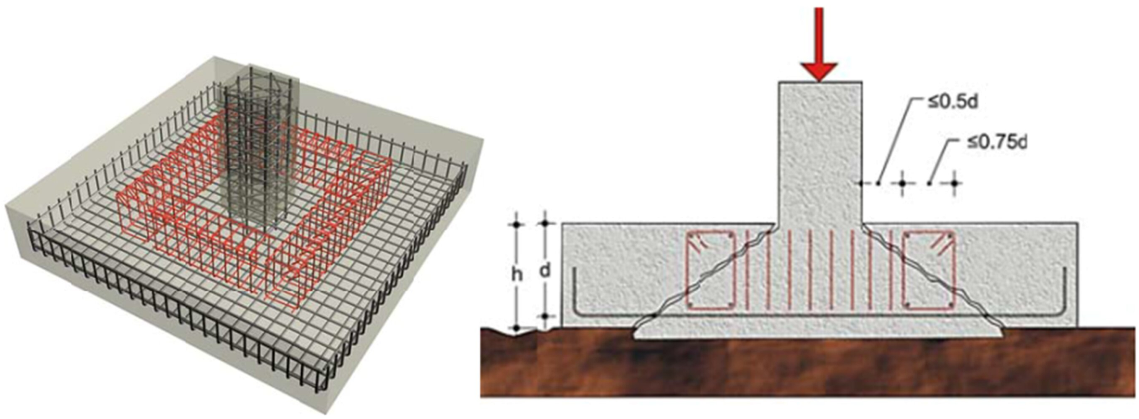

Figure 7.44: Links (stirrups) as punching shear reinforcement.

- α: the angle between shear steel and the plane of the slab (normally taken as $\alpha = 90^\circ$ in practice).
- u: the length of the outer perimeter of the zone.

Punching Shear Reinforcement Systems: Studs

- Punching shear reinforcement arranged in two perpendicular directions.
- Decon shear stud reinforcement (Figure 7.45).

(a) Shear studs in place around a column.

(b) Shear studs attached to base plate.

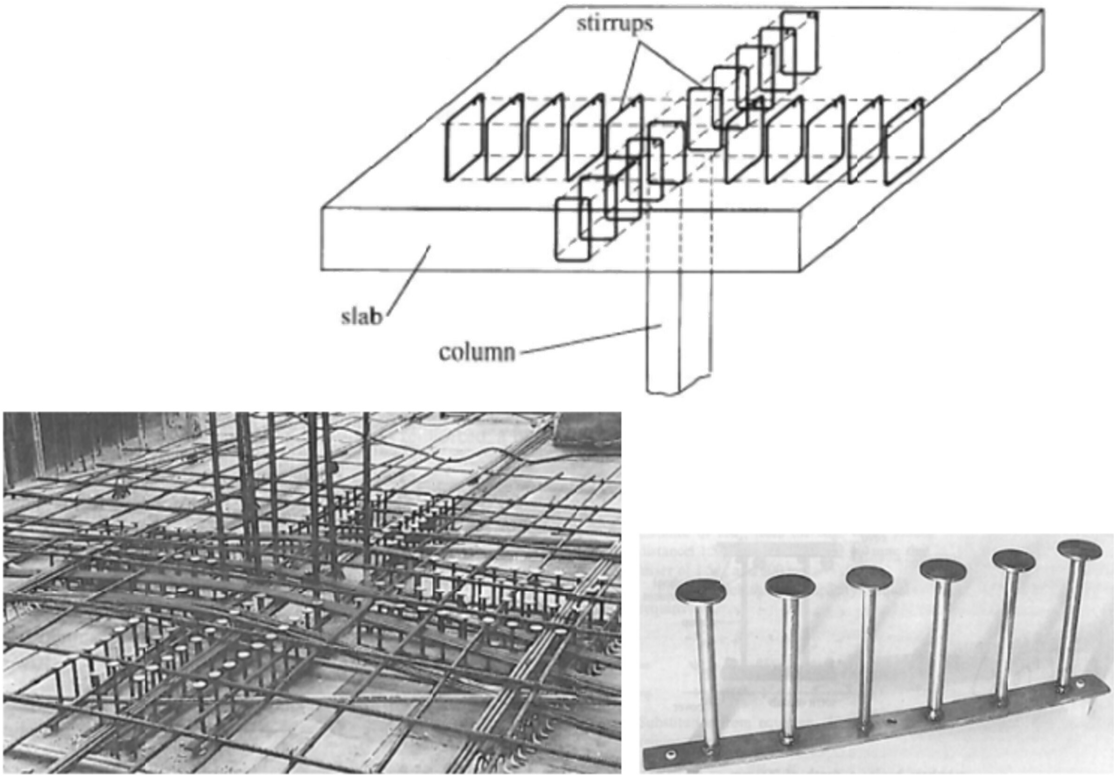

Figure 7.45: Punching shear reinforcement (a) shear stubs in place around a column, (b) shear stubs attached to base plate.

FAQs on Slabs

This is a collection of frequently asked questions (FAQs) related to design of Slabs, based on common student inquiries and clarifications provided during lectures.

Q1: In a one-way slab, why do we need reinforcement in the B-B section if there is no bending moment?

For design purposes, we assume all loads are transferred in one direction, implying no theoretical moments in the perpendicular direction (B-B section). However, in reality, secondary effects like Poisson's ratio, temperature changes, and shrinkage cause small moments. We address this by placing **distribution steel** (or redistribution steel) to prevent cracking and ensure structural integrity.

Q2: Why does the short direction of a slab require more reinforcement than the long direction?

The shorter span has higher **stiffness** than the longer span. For a given deflection at the center of the slab, the stiffer (shorter) path attracts a much larger share of the load and thus experiences greater bending moments.

Q3: In the one-way slab example, why is the end span moment $0.086FL$ instead of $0.075FL$? And why is the end moment at A zero?

The coefficient $0.086FL$ is correct because support A is a **simple support** (not a continuous support). If support A were continuous, the span moment would be $0.075FL$ and there would be a hogging moment of $-0.04FL$ at the end. Since the end is simply supported in this example, the moment at A is **zero**, and the span moment increases to $0.086FL$.

Chapter 8

Columns

Overview: This chapter covers the following topics:

- **Load Transfer and Behavior**: Columns primarily resist axial compression from vertical loads, with bending moments arising from frame continuity or lateral forces, depending on location and structural system.
- **Bracing and Role**: Braced columns handle vertical loads, relying on external elements for lateral stability, while unbraced columns resist both vertical and lateral loads through bending, increasing design complexity.
- **Slenderness and Stability**: Short columns fail by material crushing, but slender columns risk buckling due to lateral deflections, classified by effective height relative to cross-sectional dimensions.
- **Analysis Methods**: First-order theory analysis uses undeformed geometry for stable structures; advanced analysis accounts for deformed geometry and deflections, critical for slender or unbraced columns.
- **Reinforcement Principles**: Transverse reinforcement provides confinement and prevents bar buckling, which is imperative for ductility.
- **Axial Capacity**: Braced column capacity accounts for minor eccentricities and construction imperfections, adjusted from theoretical maximums.
- **Biaxial Bending**: Combines moments about two axes by amplifying the dominant moment, adjusted for axial load effects, to streamline design.
- **Shear Consideration**: Shear is typically minor but increases with axial compression, requiring additional design only when significant, similar to beam methods.
- **Practical Design Focus**: Emphasizes balancing strength, stability, and ductility using code-based guidelines, ensuring columns withstand expected loads and deformations.

8.1 Columns in Buildings

8.1.1 Vertical Loads in Regular Buildings

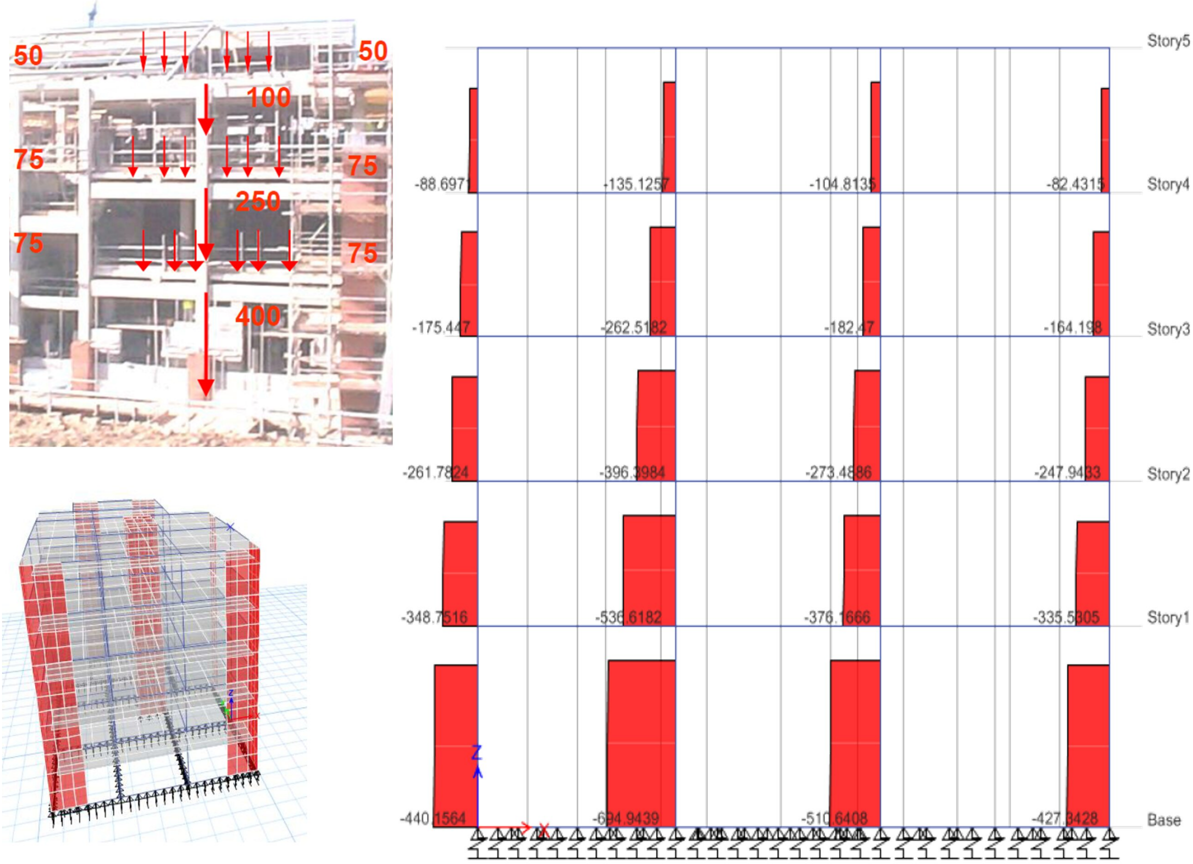

Figure 8.1: Axial force diagram for a typical column in a multistory frame building under vertical loads, illustrating the progressive increase in axial force from the top to the bottom as loads from each floor accumulate.

In regular multistory buildings, columns are critical structural elements that transfer vertical loads from slabs and beams to the foundations, primarily acting as compression members. Under gravity loading, columns predominantly experience axial forces (N), which accumulate as loads from each floor are transferred downward. As a result, the axial force in a column increases progressively from the top to the bottom of the building, see Figure 8.1.

While columns are primarily subjected to axial compression, they often must also resist bending moments (M) due to the structural continuity of the frame, frame action, or the presence of lateral loads. The magnitude of these moments depends on several factors, including the column's location within the building, the type of structural system (braced or unbraced), and the nature of the applied loads. In braced structures under vertical loads, moments in internal columns are typically small because the bracing system resists lateral forces, minimizing lateral deformation. However, external columns may experience more significant moments due to their exposure to frame action and potential eccentricities in load transfer. In unbraced structures, or when lateral loads (e.g., wind) are significant, bending moments in columns can become substantial, particularly at the base or at beam-column joints.

Bending moment diagrams (BMDs) provide valuable insight into the distribution of moments along the height of a column in a regular building. Figure 8.2 illustrates a typical BMD for a frame building, highlighting the differences between internal and external columns. Internal columns often exhibit smaller moments with a relatively uniform distribution, reflecting their primary role in resisting axial loads. In contrast, external columns may show larger moments, especially near beam-column joints, due to the frame's continuity and the transfer of shear forces from adjacent beams. The corresponding free-body diagrams in the figure further clarify the force and moment interactions at critical sections of the column, aiding in the understanding of the column's behavior under load.

In summary, columns in regular buildings under vertical loads are primarily subjected to axial compression, with the axial force increasing from the top to the bottom of the structure. Bending moments, while generally small in internal columns of braced systems, can become significant in external columns or in unbraced structures, particularly under lateral loading conditions. Understanding these internal force distributions is essential for the proper design and analysis of columns to ensure structural stability and safety.

8.1.2 Types of Columns

Columns in buildings are classified according to (Table 8.1) (i) their role in resisting lateral loads into 'braced' or 'unbraced', and (ii) their susceptibility to stability effects into 'short' or 'slender'. This classification determines their design and behavior under various loading conditions.

Table 8.1: Column classification based on slenderness ratios.

Structure Type	Short	Slender
Braced	$\frac{l_{ex}}{h} < 15$ and $\frac{l_{ey}}{b} < 15$	$\frac{l_{ex}}{h} > 15$ or $\frac{l_{ey}}{b} > 15$
Unbraced	$\frac{l_{ex}}{h} < 10$ and $\frac{l_{ey}}{b} < 10$	$\frac{l_{ex}}{h} > 10$ or $\frac{l_{ey}}{b} > 10$

Lateral Load Resistance Mechanism

In addition to vertical loads buildings are subjected to lateral loads, such as wind, or earthquake forces. The mechanism by which a structure resists these lateral loads determines the classification of its columns:

- **Braced Columns:** In braced structures, lateral loads are resisted by bracing elements (e.g., shear walls or diagonal bracing), allowing columns to primarily resist vertical loads, such as dead and imposed loads. The bracing minimizes lateral deformation, reducing the bending moments in columns.
- **Unbraced Columns:** In unbraced structures, columns resist lateral loads through bending action in addition to vertical loads. This dual role increases the bending moments and lateral deflections in the columns, requiring careful consideration of stability effects.

Figure 8.3 illustrates the difference between braced and unbraced frames, highlighting their respective mechanisms for resisting lateral loads.

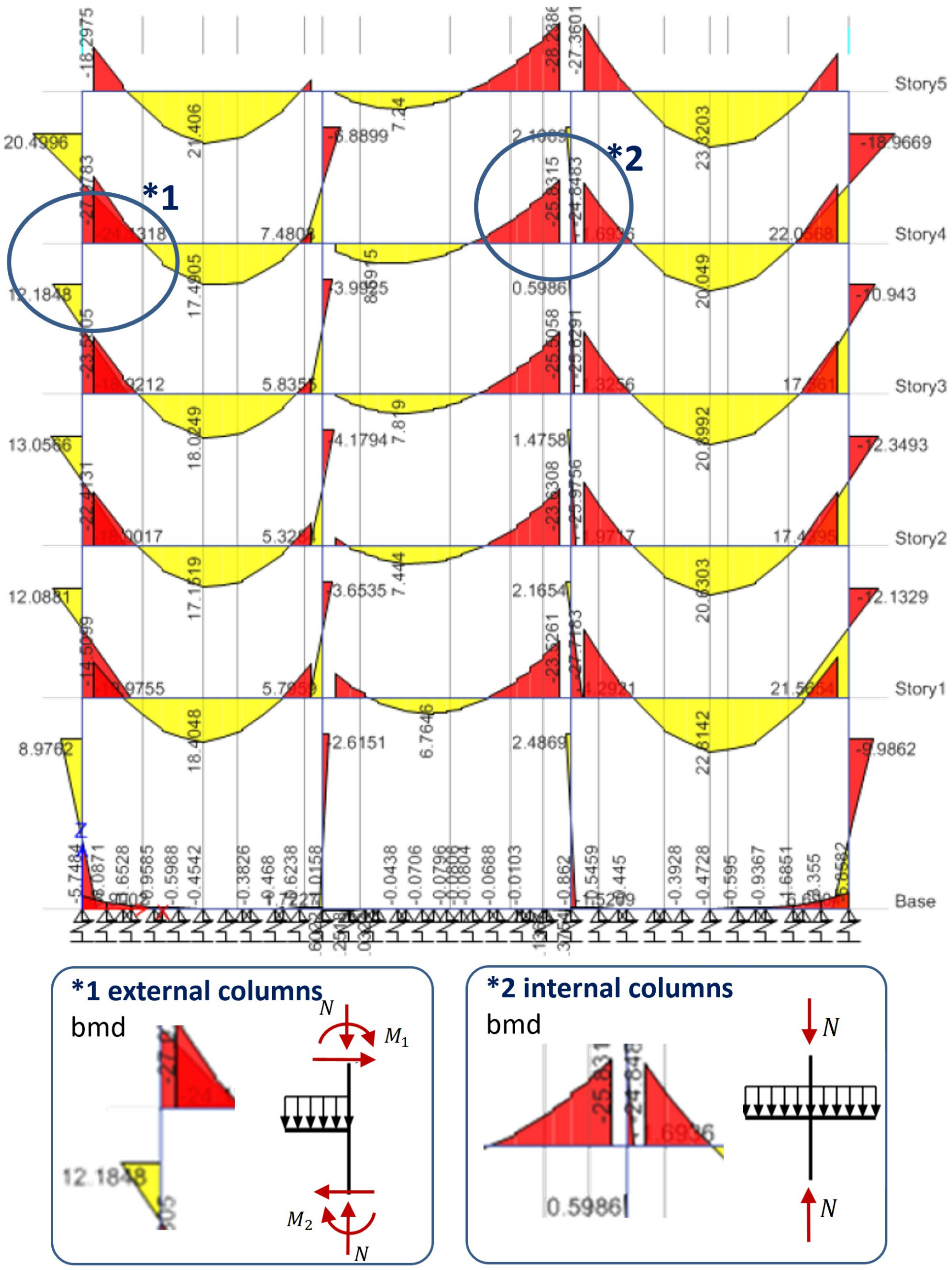

Figure 8.2: (Top) Typical bending moment diagram (BMD) for a column in a multistory frame building under vertical loads. (Bottom) Detailed view of the BMD for an internal and an external column, with corresponding simplified free-body diagrams illustrating the force and moment interactions at critical sections.

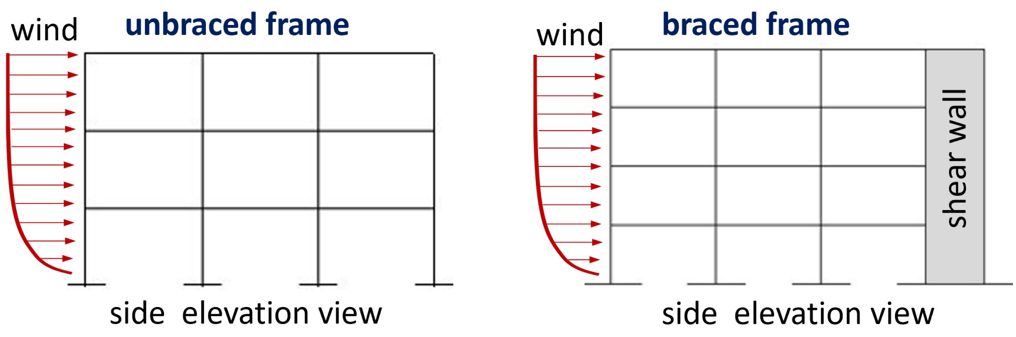

Figure 8.3: Classification of buildings according to lateral loads resistance mechanism: In unbraced frames (left) columns resist lateral loads through bending. In braced frames (right) lateral loads are resisted by bracing elements (usually shear walls or braces.

Column Slenderness

Columns in braced and unbraced structures are further categorized based on their slenderness. Table 8.1 estimates the slenderness ratio of columns via the $\frac{l_{ex}}{h}$ or $\frac{l_{ey}}{b}$ ratio.

Short (Stocky) Columns

- Short Columns are characterized by a low slenderness ratio, these exhibit minimal lateral displacement effects (typically less than 10% of first-order moments).
- Short columns fail by material crushing under combined N and M rather than instability.
- Under combined axial load and bending, failure is assessed using the $M - N$ interaction curve, as detailed in Section 2.8.

Slender Columns

- Slender Columns have a high slenderness ratio (estimated via $\frac{l_{ex}}{h}$ or $\frac{l_{ey}}{b}$ in Table 8.1), and experience significant second-order effects from lateral deflections, amplifying moments and potentially leading to buckling rather than material failure.
- Consequently, slender columns are prone to buckling due to significant lateral deflections.
  - An additional moment, $N \cdot e_{\text{add}}$, arises from lateral deflection, amplifying the primary moment.
  - Buckling occurs when N exceeds the critical Euler load (N_{crit}) (Figure 8.19).

Classification depends on the ratios of effective heights (l_{ex}, l_{ey}) to cross-sectional dimensions (h, b), as shown in Table 8.1. Effective height, accounting for end conditions, is detailed in Figure 2.49. The terms in Table 8.1 are:

- l_{ex}: Effective height for major axis bending.
- l_{ey}: Effective height for minor axis bending.
- h: Cross-sectional depth (major axis direction).
- b: Cross-sectional width (minor axis direction).

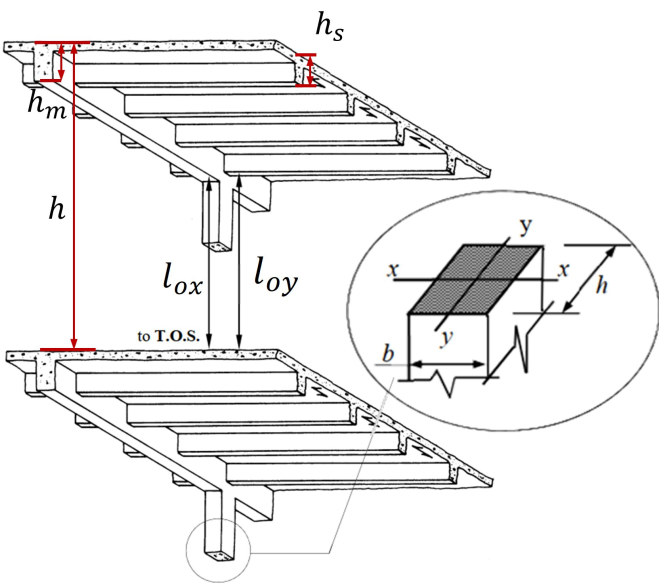

Figure 8.4: Effective height of a column with beam restraints.

Effective Height

The effective height (l_e) reflects the influence of end restraints on column stability and is calculated as:

$$l_e = \beta l_o$$

where:

- β: A coefficient based on end restraint conditions (see Table 8.2).
- l_o: Clear distance between end restraints.

For bending about the major and minor axes (see Figure 8.4):

$$l_{ox} = h - h_m, \quad l_{oy} = h - h_s$$

where:

- h: Floor-to-floor height.
- h_m: Main beam depth (major axis bending).
- h_s: Secondary beam depth (minor axis bending).

Table 8.2: Effective height coefficients (β) for braced and unbraced columns.

Top Condition	Braced Columns			Unbraced Columns		
	Bottom Condition			Bottom Condition		
	1	2	3	1	2	3
1	0.75	0.80	0.90	1.2	1.3	1.6
2	0.80	0.85	0.95	1.3	1.5	1.8
3	0.90	0.95	1.00	1.6	1.8	–
4	–	–	–	2.2	–	–

End Conditions for Table 8.2 The coefficient β varies with end conditions and bracing, as shown in Table 8.2. Braced columns typically have lower β values due to enhanced restraint, while unbraced columns have higher values, indicating greater susceptibility to lateral instability. These coefficients are critical for assessing slenderness and ensuring stability under load.

1. Monolithic connection to beams on either side, at least as deep as the column's dimension in the plane considered.
2. Monolithic connection to beams or slabs shallower than the column's dimension in the plane considered.
3. Connection to members providing nominal rotational restraint, not specifically designed for it.
4. Unrestrained against lateral movement and rotation (e.g., cantilever free end in an unbraced structure).

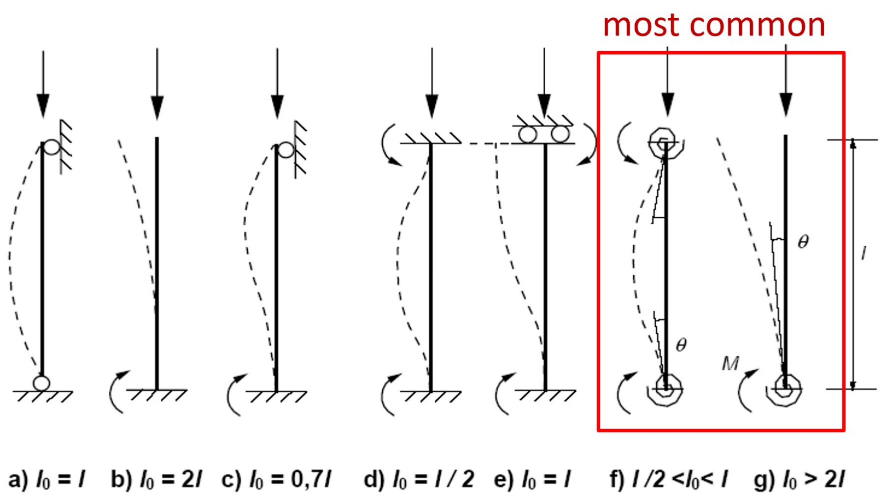

Figure 8.5: Column End Conditions

Figure 8.5 visualizes different end (top and bottom) conditions of columns and the corresponding effective lengths l_0 in each case. Note that the numerical values of Figure 8.5 are only to facilitate understanding and should not to be used instead of β values of Table 8.2.

8.1.3 Failure Modes of Columns Additional Material

Short columns, with $l_0/i < 50$, typically fail by material crushing under combined axial load N and moment M, as the concrete and reinforcement reach their compressive capacities. Here the slenderness ratio is defined as l_0/i, where l_0 is the effective length and i is the radius of gyration. In contrast, slender columns, with $l_0/i > 110$, are prone to instability failure through buckling, where geometric nonlinearity dominates. For intermediate slenderness ratios ($50 \leq l_0/i \leq 110$), mixed failure modes occur, combining material crushing and instability effects. Figure 8.6 illustrates the failure modes of columns for different slenderness.

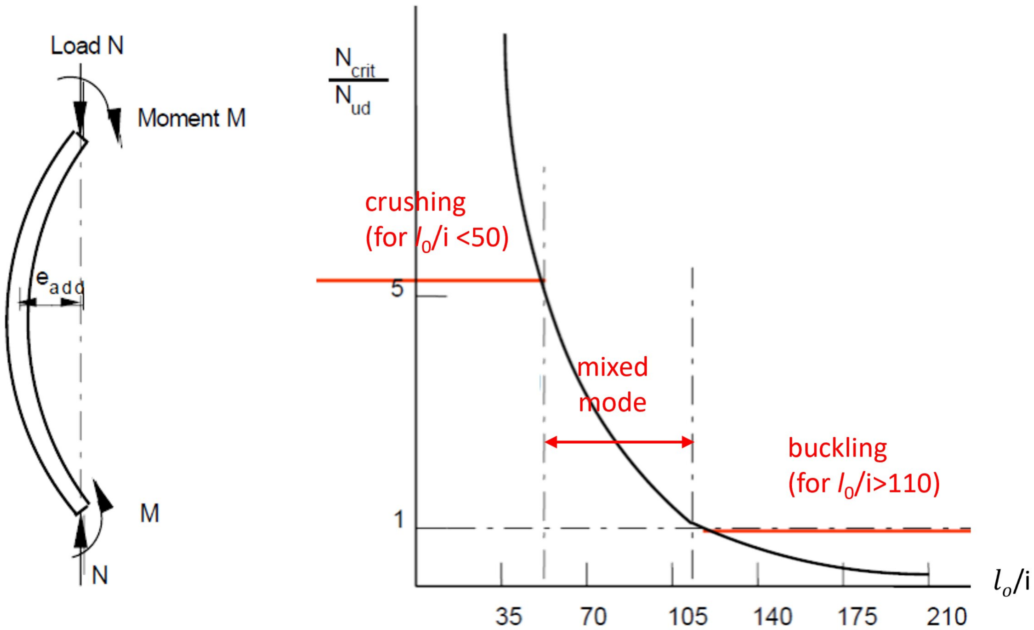

Figure 8.6: Failure modes of columns: short columns fail by material crushing, while slender columns are susceptible to buckling instability (Additional Material).

8.2 Confinement in Reinforced Concrete

Confinement is a fundamental concept in the design of reinforced concrete structures, particularly for columns and other compression members. It involves the use of transverse reinforcement, such as ties (hoops) or spirals, to provide lateral support to the concrete core. This lateral support restricts the concrete's tendency to expand (dilate) under compressive loads, which can otherwise lead to internal cracking and premature failure (Figure 8.7).

The primary benefits of confinement are increased **strength** and improved **ductility**. The effectiveness of confinement depends on the type, amount, and configuration of the transverse reinforcement. Stirrups/Ties/Hoops, typically used in rectangular or square columns, apply tensile forces primarily at their ends, resulting in localized regions of high confinement (Figure 8.7). Ties are effective in providing confinement in rectangular or square columns, but they are less efficient than spirals in circular columns. Spirals distribute the confining pressure more evenly around the column, due to their continuous nature, and hence are particularly effective in preventing dilation and providing uniform confinement.

The mechanism of confinement can be explained as follows:

- Under axial compression, concrete tends to expand laterally due to Poisson's effect.
- This lateral expansion is resisted by the transverse reinforcement, which exerts a compressive force on the concrete core.
- The restraint provided by the transverse reinforcement generates a lateral confining stress, denoted as σ_ℓ, which acts perpendicular to the direction of the axial load.
- The presence of this lateral stress transforms the state of stress in the concrete from uniaxial compression to triaxial compression.
- Triaxial compression increases the ultimate compressive strength of the concrete and allows it to sustain larger strains before failure, thereby enhancing ductility.

8.2.1 Spiral Reinforcement

To quantify the effect of confinement we examine the case of spiral reinforcement, commonly used in circular columns. It holds:

- **Lateral Confining Stress (σ_ℓ):** (Figure 8.7).

$$\sigma_\ell = \frac{2 f_{yv} A_{sv}}{d_v s}$$

  where:

  - f_{yv} is the yield strength of the spiral reinforcement,
  - A_{sv} is the cross-sectional area of the spiral,
  - d_v is the diameter of the core confined by the spiral,
  - s is the spacing of the spiral.

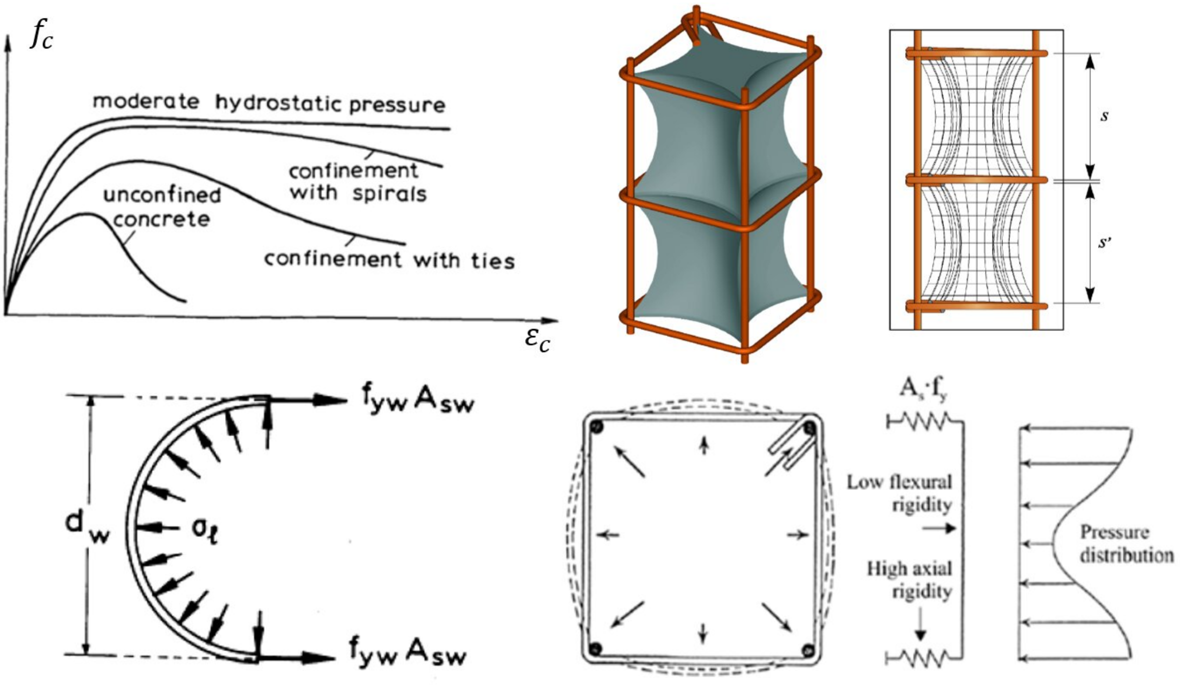

Figure 8.7: The effect of confinement in concrete: on the stress-strain behavior (top left), the confinement effectiveness of stirrups or hoops (top right and bottom right), the confining stres of spirals (bottom left).

- **Increase in Strength:**

$$f_{cc} = f_c + \lambda \sigma_\ell \Rightarrow f_{cc} = f_c \left(1 + \lambda \frac{2 f_{yv} A_{sv}}{d_v s f_c}\right) = f_c \left(1 + \frac{\lambda}{2} \omega_w\right)$$

where:

  - f_c is the compressive strength of unconfined concrete,
  - f_{cc} is the compressive strength of confined concrete,
  - λ is the confinement effectiveness factor, which accounts for the efficiency of the spiral in providing confinement. Note that for $\lambda \approx 4$, the strength increase provided by spiral reinforcement is at least twice that provided by ties, highlighting the superior confinement capability of spirals.
  - ω_w is the volumetric ratio of spiral reinforcement

$$\omega_w = \rho_w \frac{f_{yv}}{f_c} = \frac{\pi d_v A_{sv}}{\pi d_v^2 s} \cdot \frac{f_{yv}}{f_c} = \frac{4 A_{sv}}{d_v s} \cdot \frac{f_{yv}}{f_c}$$

  - ρ_w is the reinforcement ratio.

In summary, confinement through transverse reinforcement is essential for improving the performance of reinforced concrete columns under compression.

8.3 Reinforcement Detailing

Proper detailing of longitudinal and transverse reinforcement enhances the column's ability to withstand both gravity and lateral loads, prevents premature failure (e.g., buckling or crushing), and ensures adequate confinement of the concrete core, particularly in seismic or high-load conditions. The following sections outline the requirements for longitudinal and transverse reinforcement, including specific provisions for critical zones, as per the Hong Kong Concrete Code (HKCC2013) [7].

8.3.1 Longitudinal Reinforcement

In columns a minimum number of bars of longitudinal reinforcement is required (Figure 8.8):

- For rectangular columns: A minimum of 4 bars with a diameter $\geq 12\,\mathrm{mm}$.
- For circular columns: A minimum of 6 bars with a diameter $\geq 12\,\mathrm{mm}$.

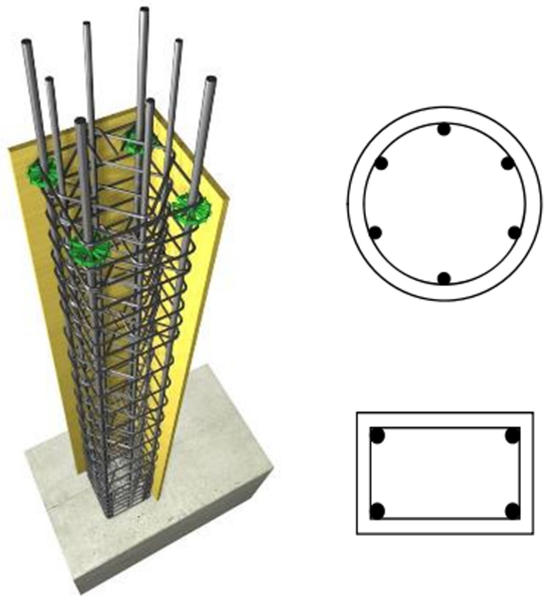

Figure 8.8: Typical detailing of longitudinal reinforcement in columns.

Reinforcement Ratios The reinforcement ratio, defined as the ratio of the area of longitudinal reinforcement (A_{sc}) to the gross cross-sectional area of the column (A_c), must be controlled to balance strength, ductility, and constructability. The general requirement is:

$$0.8\% \leq \frac{A_{sc}}{A_c} \leq 6\%$$

The lower limit ensures sufficient reinforcement to resist minimum bending moments, while the upper limit prevents congestion, which could hinder concrete compaction and lead to construction defects. For columns requiring enhanced ductility, such as in seismic design, HKCC2013 Clause 9.9.2.1 specifies a more stringent upper limit:

$$0.8\% \leq \frac{A_{sc}}{A_c} \leq 4\%$$

When the reinforcement ratio $\frac{A_{sc}}{A_c} \geq 0.8\%$, the column can resist a minimum bending moment $M_{\text{min}} = Ne_{\text{min}}$, where N is the axial force and e_{min} is the minimum eccentricity (see Figure 8.9), defined as:

$$e_{\text{min}} = \min\{0.05h, 20\,\text{mm}\}$$

Here, h is the cross-sectional dimension of the column in the direction of bending. This

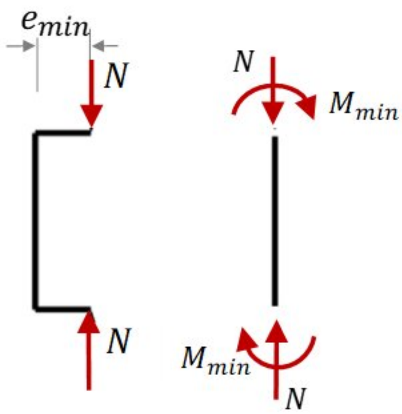

Figure 8.9: Moment from minimum eccentricity.

minimum eccentricity accounts for unintentional imperfections in construction, ensuring that the ('short') column is designed for combined axial and bending effects even under predominantly axial loading.

Splices of Column Reinforcement (Non-Seismic Details) For ductility detailing, HKCC2013 Clause 9.9.2.1(d) specifies the following requirements for lap splices (Figure 8.10):

- The center of the splice must be located within the middle quarter of the storey height (H) of the column, ensuring that the splice is away from regions of high moment and shear (typically near the column ends).
- Specifically, the center of the lap length (L.L.) should be at a height not less than $H/4$ above the floor level, where H is the storey height.

These requirements minimize the risk of splice failure by placing the lap in a region of relatively low stress, while also ensuring that the splice does not interfere with the column's ability to develop its full strength.

8.3.2 Transverse Reinforcement

Transverse reinforcement, typically in the form of links or ties, provides confinement (Figure 8.7) to the concrete core, resists shear forces, and prevents buckling of the longitudinal bars (Figure 8.13). The following requirements, based on HKCC2013 Clause 9.5.2, ensure proper confinement and stability:

- **Size (Diameter, ϕ):**

$$\phi \geq \max\{6\,\text{mm}, \frac{1}{4}\phi_{\text{largest compression bar}}\}$$

  This ensures that the links are sufficiently robust to provide confinement and restrain the longitudinal bars against buckling.

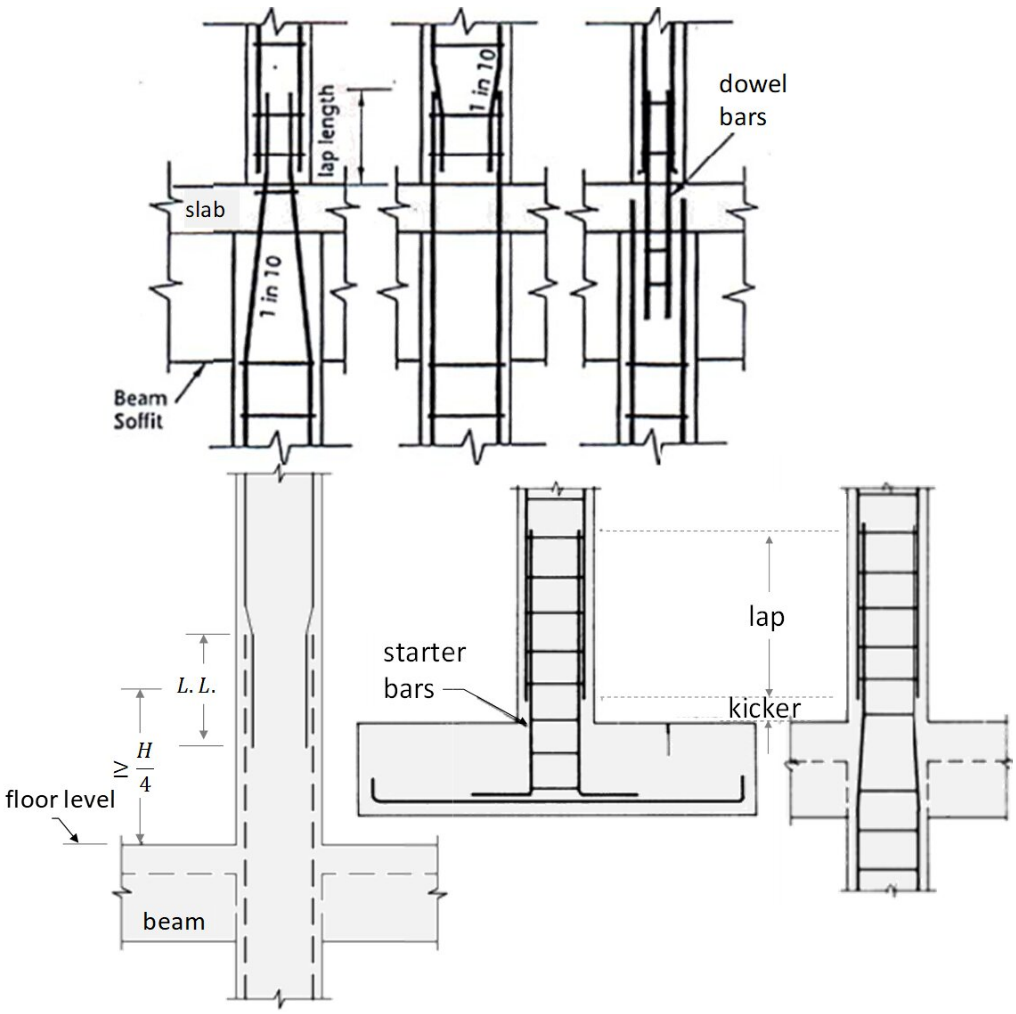

Figure 8.10: Details of splices of reinforcement in columns. Top left: splices for non-seismic details. Bottom: links in columns

- **Spacing:**

$$\text{Spacing} \leq 12 \times \phi_{\text{smallest compression bar}}$$

  Additionally, the Institution of Structural Engineers (IStructE) recommends that the link spacing should not exceed the smallest cross-sectional dimension of the column, ensuring adequate confinement throughout the column's length.

The arrangement of links must ensure proper restraint of the longitudinal bars. Every corner bar and alternate bar in the outer layer of longitudinal reinforcement should be laterally supported by a link passing around the bar, with an included angle of less than $135°$, making the bar restrained. No longitudinal bar should be farther than $150\,\text{mm}$ from a restrained bar, ensuring that all bars are adequately supported against buckling. Links must be anchored with hooks having a hook angle of at least $135°$ (HKCC2013 Clause 9.5.2.2) to prevent pullout and ensure effective confinement.

Crossties, used to provide additional restraint in larger columns, should be alternated end-for-end along the longitudinal bars to maintain symmetry and uniform confinement. Figure 8.11 illustrates the detailing of transverse reinforcement and crossties in columns.

Transverse Reinforcement in Critical Zones Critical zones in columns, such as regions near beam-column joints or at the column base, are subjected to high moments, shear

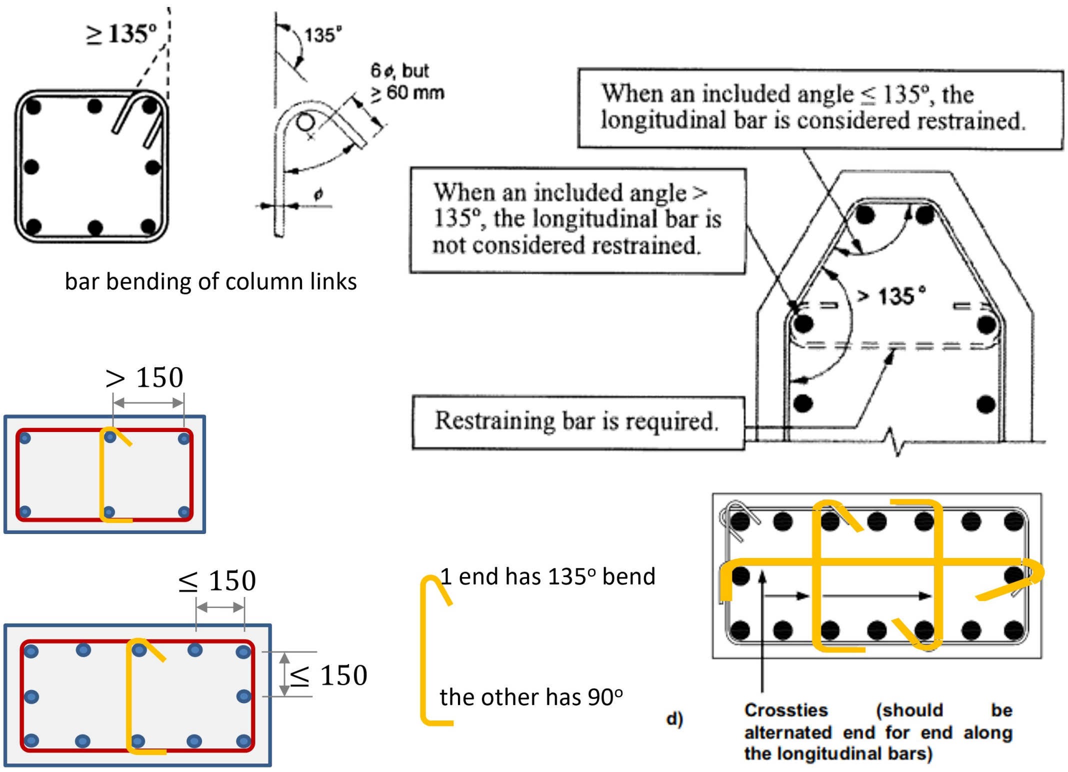

Figure 8.11: Detailing of transverse reinforcement in columns, showing the arrangement of links and crossties to provide confinement and restrain longitudinal bars.

forces, and potential plastic hinging, particularly under seismic loading. The length of the critical zone (l_o) depends on the axial load ratio $N/(A_g f_{cu})$, where A_g is the gross cross-sectional area of the column in mm^2 and f_{cu} is the characteristic compressive strength of concrete. Figure 8.12 illustrates the suggested critical zone length in a column. The following criteria define l_o, including the zone influenced by the stub effect:

- For $0 < N/(A_g f_{cu}) \leq 0.1$:

$$l_o = \max \left\{ \begin{array}{l} 1.0 \times \text{larger dimension of cross-section,} \\ \text{where moment exceeds } 0.85 \times \text{max moment,} \\ \frac{1}{6} \times \text{column clear height at floor} \end{array} \right\}$$

- For $0.1 < N/(A_g f_{cu}) \leq 0.3$:

$$l_o = \max \left\{ \begin{array}{l} 1.5 \times \text{larger dimension of cross-section,} \\ \text{where moment exceeds } 0.75 \times \text{max moment,} \\ \frac{1}{6} \times \text{column clear height at floor} \end{array} \right\}$$

- For $0.3 < N/(A_g f_{cu}) \leq 0.6$:

$$l_o = \max \left\{ \begin{array}{l} 2.0 \times \text{larger dimension of cross-section,} \\ \text{where moment exceeds } 0.65 \times \text{max moment,} \\ \frac{1}{6} \times \text{column clear height at floor} \end{array} \right\}$$

These criteria ensure that the critical zone is sufficiently long to account for regions of high stress and potential plastic deformation, with the length increasing as the axial load ratio increases due to the greater need for confinement.

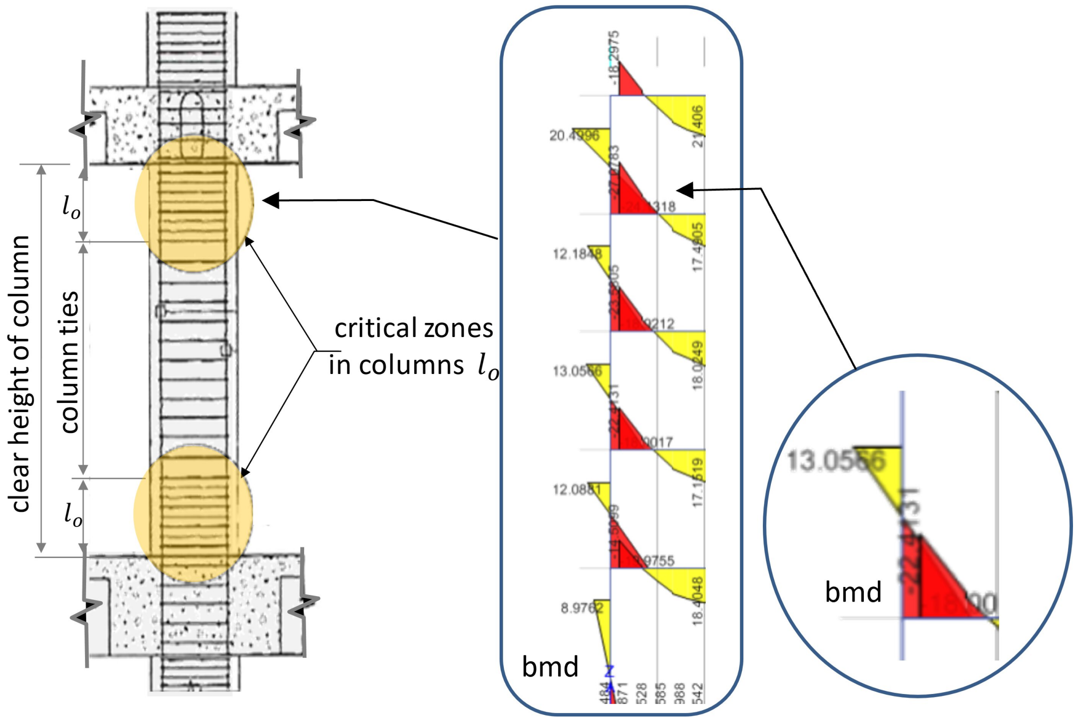

Figure 8.12: Suggested critical zone length (l_o) in a column, highlighting regions of high moment where enhanced transverse reinforcement is required.

Within the critical zone, the following requirements apply to ensure enhanced confinement and ductility:

- **Size within Critical Zone:**

$$\phi \geq \text{max}\{10\,\text{mm}, \frac{1}{4}\phi_{\text{largest longitudinal bar}}\}$$

  The larger minimum diameter reflects the need for stronger links to provide adequate confinement in high-stress regions.

- **Spacing within Critical Zone (Rectangular/Polygonal Columns):** Centre-to-centre spacing $\leq \text{min}\{150\,\text{mm}, 8 \times \phi_{\text{longitudinal bar to be restrained}}\}$ The arrangement of links or ties must comply with one of the following:
  - Each longitudinal bar or bundle is supported by a link passing around it, ensuring full restraint.
  - Every corner bar and alternate bar in the outer layer is supported by a link, with no bar in the compression zone farther than $\text{min}\{10 \times \phi_{\text{link}}, 125\,\text{mm}\}$ from a restrained bar, ensuring adequate confinement in the compression zone.

- **Spacing within Critical Zone (Circular Columns):**

$$\text{Centre-to-centre spacing} \leq \text{min}\{150\,\text{mm}, 8 \times \phi_{\text{longitudinal bar to be restrained}}\}$$

  The same spacing requirement applies to circular columns, ensuring uniform confinement around the perimeter.

- **Anchorage:** Links and ties must be anchored with hooks having a hook angle of 135°, ensuring secure anchorage and effective confinement under cyclic loading.

These provisions for critical zones enhance the column's ability to develop plastic hinges, resist shear forces, and maintain stability under extreme loading conditions, such as earthquakes.

8.4 Braced Axially-Loaded Columns

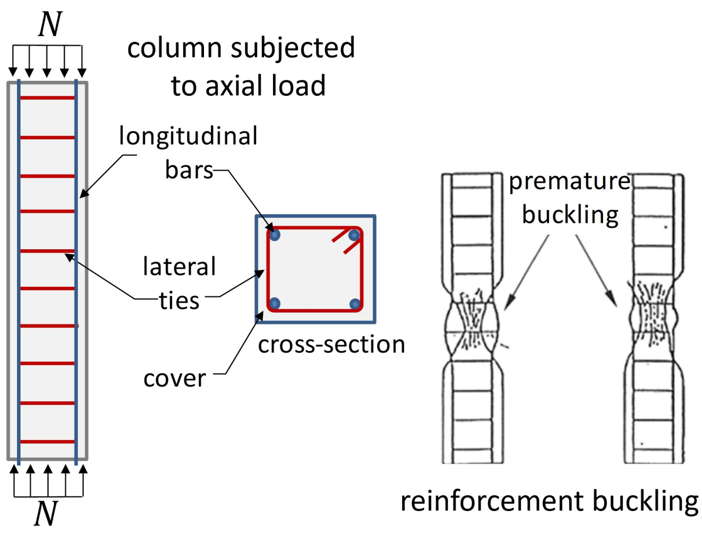

Figure 8.13: Short columns.

Column section analysis is different from beams and slabs due to the presence of the axial load (usually neglected in beams). The loading conditions of columns supporting a symmetrical and very rigid structure (braced structure), often approximate pure axial loading; i.e., the bending moment is negligible.

Typical modes of failure for such columns are crushing of concrete, potentially followed by buckling of longitudinal bars. To prevent premature buckling of longitudinal bars:

- Link size should be adequate.
- Shear links should be sufficiently closely spaced.

Ultimate Load Capacity

Recall that for pure axial compression, the ultimate design load is:

$$N_u = 0.45 f_{cu} A_c + 0.87 f_y A_{sc}$$

where A_c is the gross-sectional area and A_{sc} is the total longitudinal steel area. Note that in some design codes the net concrete area $A_{nc} = bh - A_{sc}$ is used instead of A_c in the calculation of the ultimate design load.

In design according to HKCC2013 (Clause 6.2.1.4) to allow for eccentricity of loading due to construction tolerances, the capacity for axial load is limited to about 90% of:

$$N = 0.40 f_{cu} A_c + 0.75 f_y A_{sc}$$

where:

- A_c: Gross cross-sectional area of concrete of the column
- A_{sc}: Area of the longitudinal reinforcement (total area, not tension steel only)

8.4.1 Example #1: Axially Loaded Short Column (Design)

A short-braced column must carry a design axial load of $3200\,\mathrm{kN}$. The column is $400\,\mathrm{mm}$ square in section. Material strengths: grade 35 concrete and grade 500 reinforcement. Determine the steel area required for the longitudinal reinforcement and select suitable bars.

Solution

For a braced axially-loaded column, substitute into the expression for the ultimate axial load:

$$N = 0.40 f_{cu} A_c + 0.75 f_y A_{sc} \rightarrow 3200 \cdot 10^3 = 0.4 \cdot 35 \cdot 400^2 + 0.75 \cdot 500 A_{sc} \rightarrow$$

$$A_{sc} = 2560\,\mathrm{mm}^2$$

Provide 4T25 + 4T16 to give a steel area of $2767\,\mathrm{mm}^2$. Check:

$$\frac{A_{sc}}{A_c} = \frac{2767}{400^2} = 1.72\% < 4\% \quad \text{(satisfactory)}$$

8.4.2 Example #2: Axially Loaded Short Column (Checking)

A short-braced column, $300\,\mathrm{mm}$ square in section, is reinforced with 4T25 bars. Material strengths: grade 40 concrete and grade 500 reinforcement. Determine the ultimate axial load the column can carry and the pitch and diameter of the links required.

Solution

For 4T25:

$$A_{sc} = 1963\,\mathrm{mm}^2, \quad A_c = 300 \cdot 300 = 90 \cdot 10^3\,\mathrm{mm}^2$$

The ultimate axial load:

$$N = (0.4 \cdot 40 \cdot 90000 + 0.75 \cdot 1963 \cdot 500) \cdot 10^{-3} = 2176.1\,\mathrm{kN}$$

For ductility detailing [HKCC2013, Clause 9.9.2.2]:

$$\frac{N}{A_g f_{cu}} = \frac{2176.1 \cdot 10^3}{300 \cdot 300 \cdot 40} = 0.60$$

Hence, the critical zone length is:

$$l_o = 2 \cdot 300 = 600\,\mathrm{mm}$$

Within critical zones:

- Link size: $\geq \max\{10\,\mathrm{mm}, \frac{1}{4}\phi_{\text{largest longitudinal bar}} = \frac{1}{4} \cdot 25 = 6.25\,\mathrm{mm}\} = 10\,\mathrm{mm}$
- Spacing: $\leq \min\{150\,\mathrm{mm}, 8 \cdot \phi_{\text{longitudinal bar to be restrained}} = 8 \cdot 25 = 200\,\mathrm{mm}\} = 150\,\mathrm{mm}$

Within non-critical zones:

- Link size: $\geq \max\{6\,\mathrm{mm}, \frac{1}{4}\phi_{\text{largest longitudinal bar}} = 6.25\,\mathrm{mm}\} \rightarrow 8\,\mathrm{mm}$
- Spacing: $\leq \min\{12 \cdot \phi_{\text{longitudinal bar to be restrained}} = 12 \cdot 25 = 300\,\mathrm{mm}\} = 300\,\mathrm{mm}$

Provide R10@150 within critical zones and R8@300 within non-critical zones.

8.4.3 Short Column Subjected to Axial Load and Moment

Section 2.8 discusses the analysis and design of cross-sections under combined axial load (N) and bending moment (M).

Recall that,

- Design of column cross-sections assumes symmetric reinforcement ($A_s = A'_s = A_{sc}/2$).
- Design codes provide $M - N$ interaction diagrams (charts) in dimensionless form, plotting axial load ratio ($\frac{N}{bhf_{cu}}$) against moment ratio ($\frac{M}{bh^2 f_{cu}}$), with contours of reinforcement ratio ($\frac{A_{sc} f_y}{bhf_{cu}}$) to determine A_{sc}.
- Design charts depend on steel yield strength (f_y) and geometry (d/h). Interpolation between charts (e.g., $d/h = 0.80$ and 0.85) is used if an exact match is unavailable, with sufficient accuracy for practical design.

The design procedure established in Section 2.8 is directly applicable to the design of short braced columns, as the following design example illustrates.

Example: Design of a Short Braced Column

Design of a short braced column section to determine the total amount of steel reinforcement (A_{sc}) required.

Given Data

- Section dimensions: $300 \times 300\,\mathrm{mm}$, with $\frac{d}{h} = 0.85$,
- Design axial load: $N = 1480\,\mathrm{kN}$,
- Design moment: $M = 54\,\mathrm{kNm}$,
- Material properties: $f_{cu} = 30\,\mathrm{N/mm^2}$, $f_y = 500\,\mathrm{N/mm^2}$,
- Assumed reinforcement: $25\,\mathrm{mm}$ diameter bars, $10\,\mathrm{mm}$ diameter links, $25\,\mathrm{mm}$ cover.

Solution steps:

1. Effective depth:

$$d = h - c - \phi_{\text{link}} - \frac{\phi}{2} = 300 - 25 - 10 - 12.5 = 252.5\,\mathrm{mm} \approx 253\,\mathrm{mm}$$

$$\frac{d}{h} = \frac{253}{300} \approx 0.85$$

2. Axial Load Ratio:

$$\frac{N}{bhf_{cu}} = \frac{1480 \cdot 10^3}{300 \cdot 300 \cdot 30} = 0.548 \approx 0.55$$

3. Moment Ratio:

$$\frac{M}{bh^2 f_{cu}} = \frac{54 \cdot 10^6}{300 \cdot 300^2 \cdot 30} = 0.0667 \approx 0.067$$

4. Using the design chart of Figure 2.46 ($\frac{d}{h} = 0.85$), the point (0.548, 0.067) lies near the contour 0.32, thus:

$$\frac{A_{sc} f_y}{bhf_{cu}} \approx 0.32 \rightarrow A_{sc} = 0.32 \cdot \frac{300 \cdot 300 \cdot 30}{500} = 1728\,\mathrm{mm}^2$$

Provide 4T25 (i.e., 4 bars of $25\,\mathrm{mm}$ diameter) $A_{sc} = 1963\,\mathrm{mm}^2$, placed symmetrically (2 bars at the top, 2 at the bottom). Check:

$$\rho = \frac{A_{sc}}{bh} = \frac{1963}{300 \cdot 300} = 0.0218\,(2.18\% < 4\%,\ \text{satisfactory})$$

8.4.4 Design for Shear

In general, shear forces in columns are small and may be neglected. Detailing of transverse reinforcement in columns should follow the requirements given. Shear design should be carried out if the shear force is relatively large. The procedure for designing shear reinforcement is the same as that for beams. For a section subjected to a combination of shear and axial compression, the design concrete shear stress is modified as:

$$v_c' = v_c + 0.75 \frac{N}{A_c} \frac{Vd}{M} = v_c + 0.75 \frac{N/A_c}{a_v/d} \tag{8.1}$$

where:

- $\frac{V_d}{M}$ should not be taken >1.0, and $v_c' \le \min\{0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm}^2\}$
- $\frac{N}{A_c}$: Concrete axial compression
- $\frac{a_v}{d} = \frac{M}{Vd}$: Shear span-depth ratio, ≥ 1.0

Hence, the shear capacity of concrete increases with axial compression, but the rate of increase decreases with an increase in the shear span-depth ratio.

8.4.5 Column Section with Biaxial Bending

Columns may be subjected to biaxial bending, with design moments M_x and M_y about the x- and y-axes, respectively. The procedure for constructing interaction diagrams for uniaxial bending and axial force can be extended to the biaxial case. However,biaxial bending requires the construction of three-dimensional (Figure 8.14) instead of two-dimensional interaction surfaces involving moments M_y, M_x, and axial force N. Three-dimensional surfaces are generally inconvenient for practical design. Therefore, it is common to plot two-dimensional representations of these surfaces, where each plot corresponds to a specific axial load level N. Each design chart is valid for a specific cross-section type, reinforcement pattern, cover-to-depth ratio, and steel grade. In practice, interpolations are often required with respect to both the normalized axial force ν_d and the total steel ratio ω_{tot}.

Figure 8.15 presents a sample design chart of a reinforced concrete column with concrete grade C12-C50 and steel grade B500C, subjected to biaxial bending. The design chart of Figure 8.15 plots contours of ω_{tot} levels, where:

$$\omega_{\text{tot}} = A_{sc} \cdot \frac{f_{\text{yd}}/f_{\text{cd}}}{b \cdot h}$$

The following parameters are involved:

- **Material Properties:**

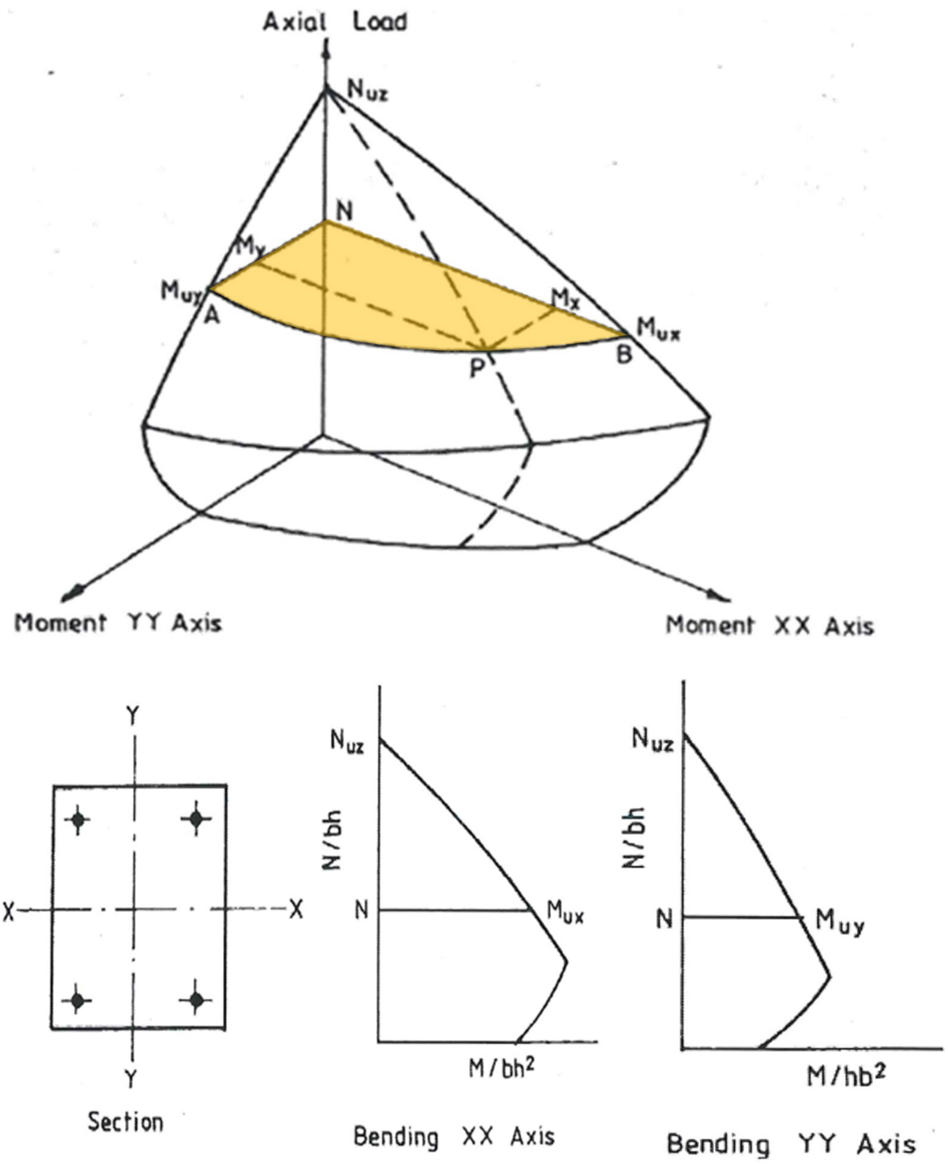

Figure 8.14: Failure surface: M_x-M_y-N interaction for biaxial bending and axial load.

- Concrete: $\varepsilon_{c2} = 0.2\%$, $\varepsilon_{cu2} = 0.35\%$, $n = 2$,
- Steel (B500C): $\delta_s = 1.15$, $k = 1.00$, $\varepsilon_{ud} = \infty$.

- **Geometric Ratios:**

$$\frac{b_1}{b} = \frac{h_1}{h} = 0.10$$

- **Normalized Design Parameters:**
  - Axial load ratio: $\nu = \frac{N_d}{(b \cdot h \cdot f_{cd})}$,
  - Bending moment ratio x-axis: $\mu_x = |M_{xd}|/(b \cdot h^2 \cdot f_{cd})$
  - Bending moment ratio y-axis: $\mu_y = |M_{yd}|/(b \cdot h^2 \cdot f_{cd})$
  - Bending moment ratios: $\mu_1 = \max(\mu_x, \mu_y)$, $\mu_2 = \min(\mu_x, \mu_y)$
  - Required steel area $A_{sc} = \omega_{\text{tot}} \cdot b \cdot h \cdot \frac{f_{cd}}{f_{yd}}$

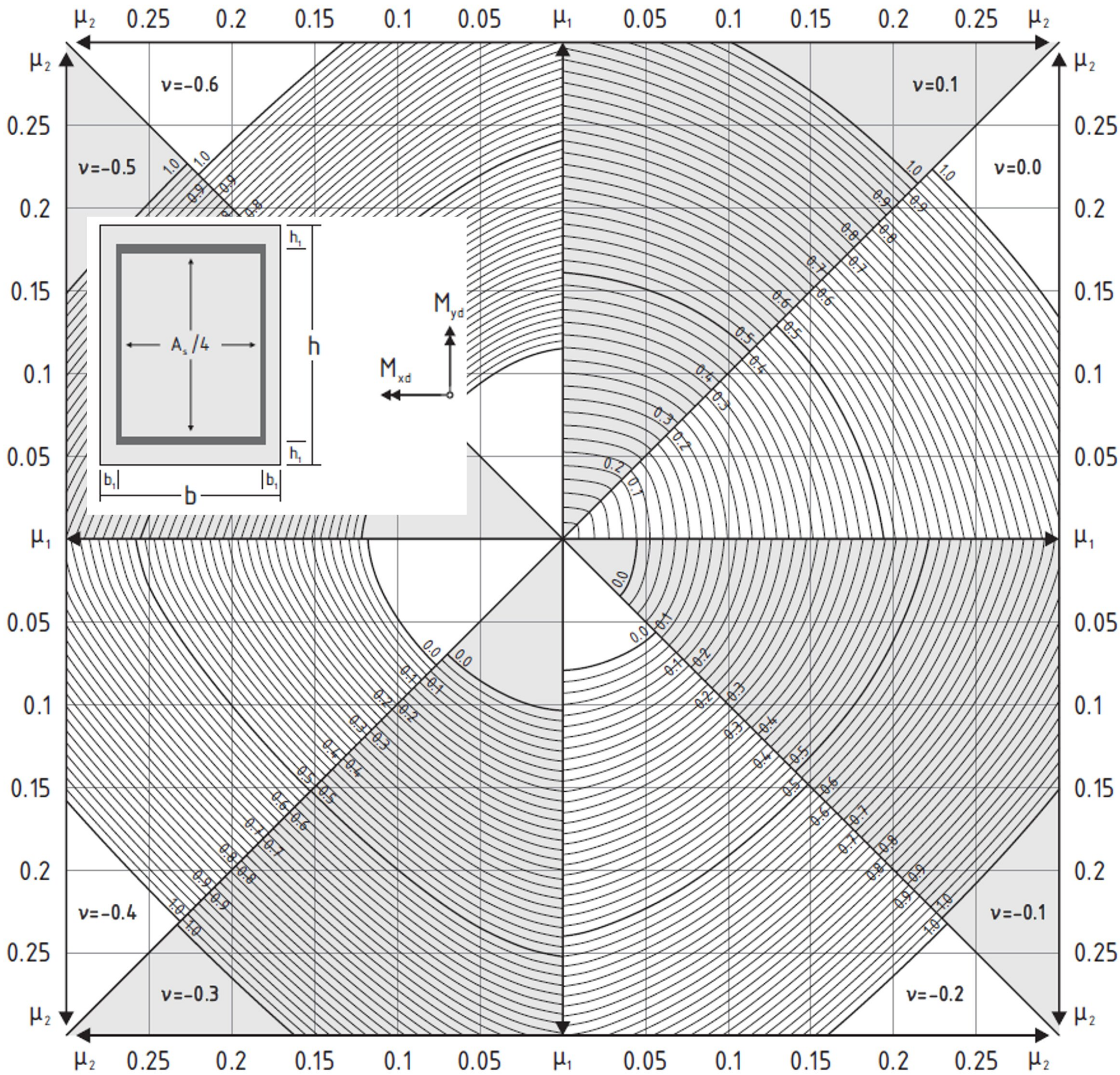

Figure 8.15: Design charts for columns subjected to biaxial bending and axial force for distributed reinforcement (Papanikolaou & Sextos (2016) [17]).

8.4.6 Code Design Method [HKCC2013: Clause 6.2.1.4]

A simplified approach for biaxial bending in symmetrically reinforced rectangular sections is to convert the problem to an equivalent uniaxial bending problem with increased moments. To this end, for ultimate load N and moments M_x, M_y, amplify the dominant moment:

1. If $\frac{M_x}{h'} \geq \frac{M_y}{b'}$: $M'_x = M_x + \beta \frac{h'}{b'} M_y$ (x-x axis governs).
2. If $\frac{M_x}{h'} < \frac{M_y}{b'}$: $M'_y = M_y + \beta \frac{b'}{h'} M_x$ (y-y axis governs).

Coefficient β from Table 8.3 adjusts for axial load effects.

Table 8.3: Values of β for biaxial bending based on axial load ratio.

$N/(bhf_{cu})$	0	0.1	0.2	0.3	0.4	0.5	≥ 0.6
β	1.00	0.88	0.77	0.65	0.53	0.42	0.30

8.4.7 Example: Design of a Column under Biaxial Bending

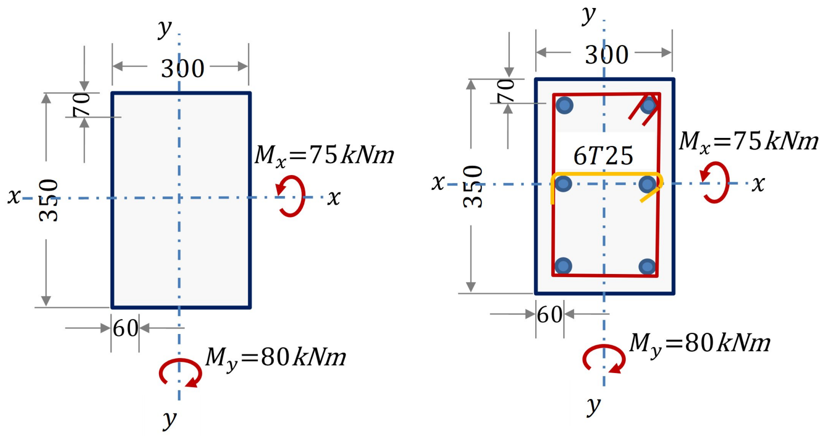

Figure 8.16: Reinforced column cross-section for biaxial bending.

Design a column section subjected to biaxial bending to determine the total amount of steel reinforcement (A_{sc}) required, **given**:

- Section dimensions: $h = 350\,\text{mm}$, $b = 300\,\text{mm}$, effective depths: $h' = 280\,\text{mm}$, $b' = 240\,\text{mm}$,
- Design axial load: $N = 1200\,\text{kN}$,
- Design moments: $M_x = 75\,\text{kNm}$, $M_y = 80\,\text{kNm}$,
- Material properties: $f_{cu} = 30\,\text{N/mm}^2$, $f_y = 500\,\text{N/mm}^2$.

Solution steps

1. Control check for governing axis:

$$\frac{M_x}{h'} = \frac{75}{280} = 0.268 < \frac{M_y}{b'} = \frac{80}{240} = 0.333\,\text{(y-y axis governs)}$$

2. Axial Load Ratio:

$$\frac{N}{bhf_{cu}} = \frac{1200 \cdot 10^3}{300 \cdot 350 \cdot 30} = 0.381 \approx 0.38$$

3. Effective moment (using $\beta = 0.55$, interpolated):

$$M'_y = M_y + \beta \cdot \frac{b'}{h'} \cdot M_x = 80 + 0.55 \cdot \frac{240}{280} \cdot 75 = 115.4\,\text{kNm}$$

4. Design parameters:

$$\frac{M'_y}{hb^2 f_{cu}} = \frac{115.4 \cdot 10^6}{350 \cdot 300^2 \cdot 30} = 0.122$$

5. Using the design chart ($\frac{d}{h} = 0.85$), at (0.38, 0.122), the contour is approximately 0.34:

$$\frac{A_{sc} f_y}{bhf_{cu}} \approx 0.34 \rightarrow A_{sc} = 0.34 \cdot \frac{300 \cdot 350 \cdot 30}{500} = 2142\,\text{mm}^2$$

Provide 6T25 (i.e., 6 bars of $25\,\text{mm}$ diameter) $A_{sc} = 2945\,\text{mm}^2$. Check:

$$\rho = \frac{A_{sc}}{bh} = \frac{2945}{300 \cdot 350} = 0.0280\,(2.80\% < 4\%,\ \text{satisfactory})$$

8.5 Slender Columns

8.5.1 1st and 2nd Order Theories

The analysis of columns under load, or structures in general, can be performed using either 1st order or 2nd order theory, depending on the consideration of structural deformations:

- **1st Order Theory:** Equilibrium conditions are applied to the original, undeformed geometry of the structure. This approach neglects the effects of deformations and is suitable for structures where deflections are small.
- **2nd Order Theory:** Equilibrium conditions are applied to the deformed geometry of the structure, accounting for additional moments caused by lateral displacements (usually referred to as P-δ effects). Figure 8.17 compares the application of first and second order theories, illustrating their impact on the analysis of column behavior. In the case of Figure 8.17 this additional moment is the term $P_y \cdot u$.

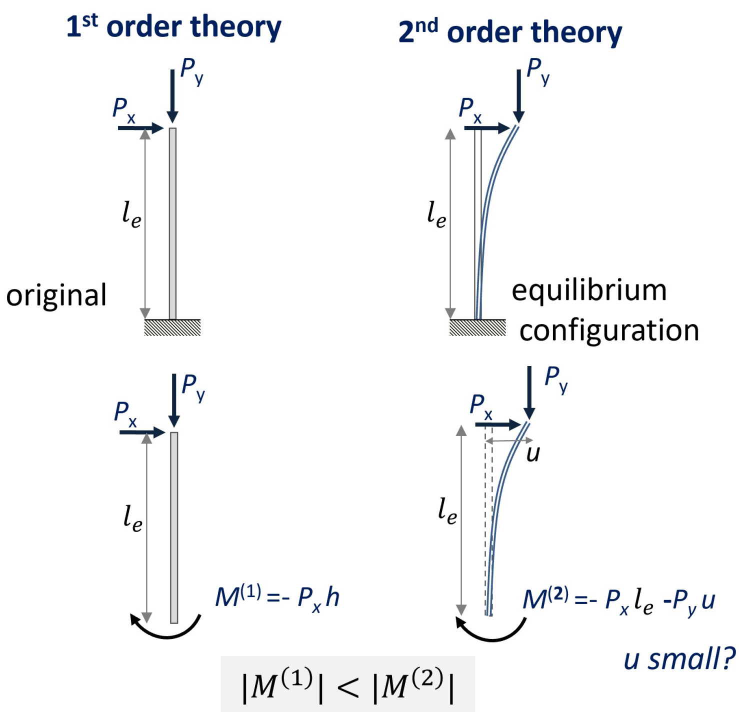

Figure 8.17: (Left) 1st order theory, where equilibrium conditions are applied to the original (undeformed) structure. (Right) 2nd order theory, where equilibrium conditions are applied to the deformed structure, accounting for additional moments due to lateral displacements.

8.5.2 Slenderness Ratio Limits

Columns in both braced and unbraced reinforced concrete structures are classified based on their slenderness ratio λ, as slender when their slenderness ratios are (Table 8.1):

$$\frac{l_{ex}}{h} > 15 \text{ or } \frac{l_{ey}}{b} > 15$$

if they are braced, and

$$\frac{l_{ex}}{h} > 10 \text{ or } \frac{l_{ey}}{b} > 10$$

if they are unbraced.

Recall that the effective height (l_e) of a column is $l_e = \beta l_o$ where coefficient β depends on the degree of end restraints (see Table 8.2), and l_o is the clear distance between end restraints, which is subjected to the following limitations:

- $l_o \leq 60$ times the minimum thickness of a column.
- If one end of an unbraced column is unrestrained (e.g., a cantilever column):

$$l_o = \frac{100b^2}{h} \leq 60b$$

Slender columns are susceptible to second-order effects.

8.5.3 Design Moment [HKCC2013: Clause 6.2.1.3]

A slender column should be designed for an ultimate axial load plus a design moment, which includes end moments and an **additional moment** M_{add} caused by the **additional eccentricity** a_u (Figure 8.18), equal to the deflection of the column under ultimate condition, due to the combined effect of N, M_1, and M_2.

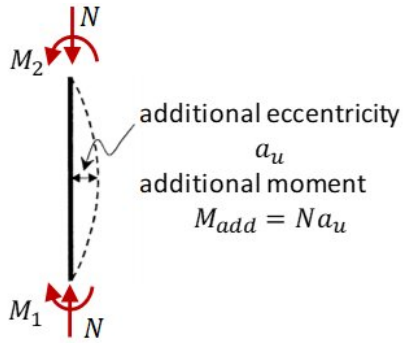

Figure 8.18: Additional moment of braced slender columns

The additional moment M_{add} is calculated by:

$$M_{\text{add}} = N a_u \tag{8.2}$$

For a rectangular or circular column:

$$a_u = \beta_a K h \tag{8.3}$$

where the slenderness factor β_a is given by:

$$\beta_a = \frac{1}{2000}\left(\frac{l_e}{b}\right)^2 \tag{8.4}$$

in which b is the smaller dimension of the column section (Clause 6.2.1.3 (f) for biaxial bending).

K is an optional reduction factor that corrects the deflection to allow for the influence of axial load. The factor K can be read from column design charts and is derived from:

$$K = \frac{N_{uz} - N}{N_{uz} - N_{\text{bal}}} \leq 1 \tag{8.5}$$

where:

- N = design ultimate axial load,
- N_{uz} = design ultimate capacity under axial load only

$$N_{uz} = 0.45f_{cu}A_{nc} + 0.87f_yA_{sc}$$

with:

- A_{nc} = net cross-sectional area of concrete in a column,
- N_{bal} = the axial load at balanced failure, which may be taken as:
- $N_{\text{bal}} = 0.25f_{cu}bd$ for symmetrically reinforced rectangular sections.

To calculate K, A_{sc} must be known, requiring a trial-and-error approach or an **iterative procedure**, taking an initial value of $K = 1.0$. Alternatively, assuming $K = 1.0$ is always conservative.

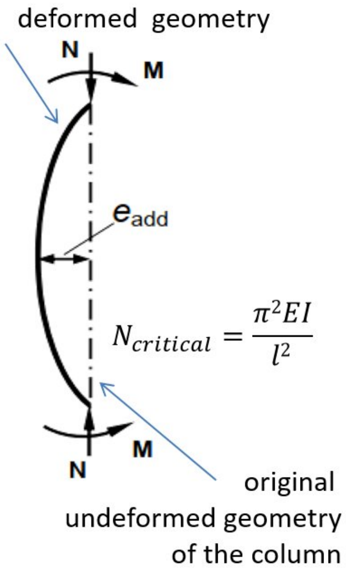

Figure 8.19: Euler critical load for a pin-ended strut.

8.5.4 Braced Slender Columns

Typical bending moment diagrams are shown in [HKCC2013: Figure 6.16] [3, 16]. The initial moment M_i in a column before allowance for additional moments arising from slenderness is:

$$M_i = 0.4M_1 + 0.6M_2 \geq 0.4M_2 \tag{8.6}$$

where:

- M_1 = the smaller initial end moment due to design ultimate loads, taken as negative if the column is bent in double curvature,
- M_2 = the larger initial end moment due to design ultimate loads, always taken as positive.

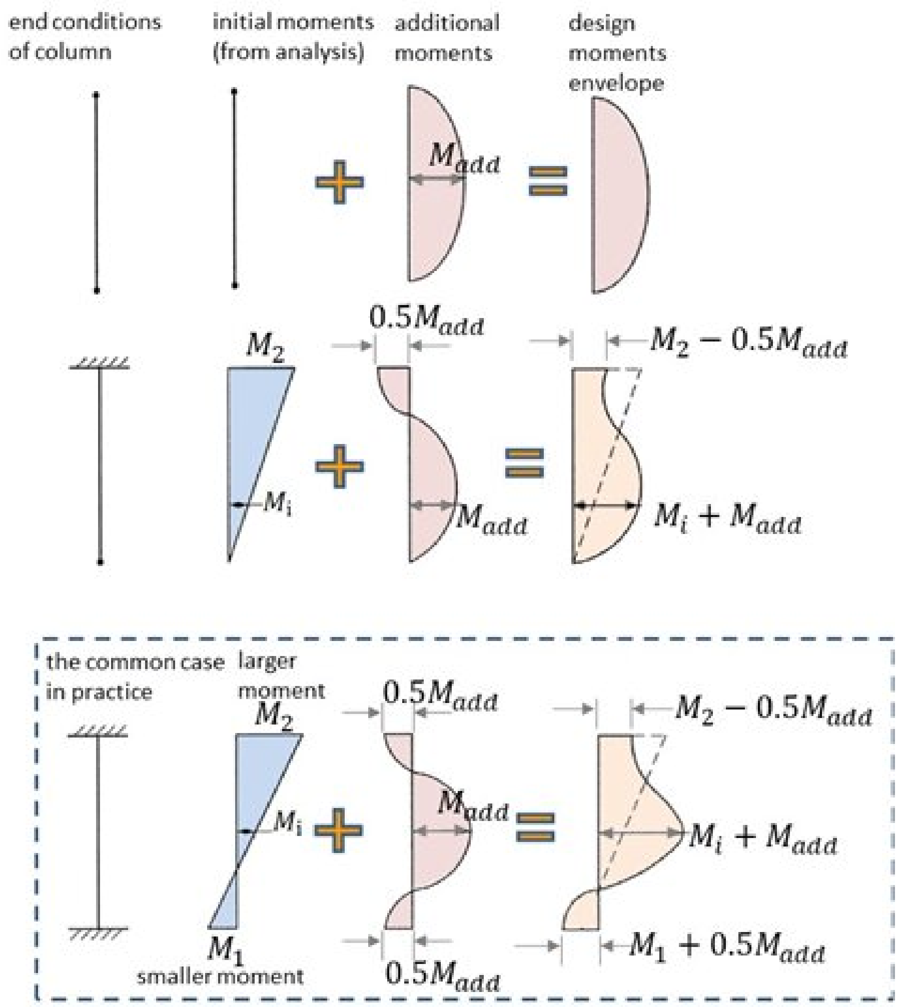

Figure 8.20: Design moments of braced slender columns

The column should be designed for its ultimate axial load N together with the maximum design moment M_d, where:

$$M_d = \max\{M_2, M_i + M_{\text{add}}, M_1 + 0.5M_{\text{add}}, Ne_{\text{min}}\} \tag{8.7}$$

with:

- $M_{\text{add}} = Na_u$ = additional moment,
- $e_{\text{min}} = \min\{0.05h, 20\,\text{mm}\}$ = design minimum eccentricity,
- h = dimension in the plane of bending.

8.5.5 Example: Design of a Slender Braced Column

A braced column of 300×450 mm cross-section resists at the ultimate limit state an axial load of $1700\,\text{kN}$ and end moments of $70\,\text{kNm}$ and $10\,\text{kNm}$, causing double curvature about the minor axis x-x.

- The column's effective heights are: $l_{ex} = 6.75$ m, $l_{ey} = 8.00$ m,
- Material grades: $f_{cu} = 30\,\text{N/mm}^2$, $f_y = 500\,\text{N/mm}^2$.

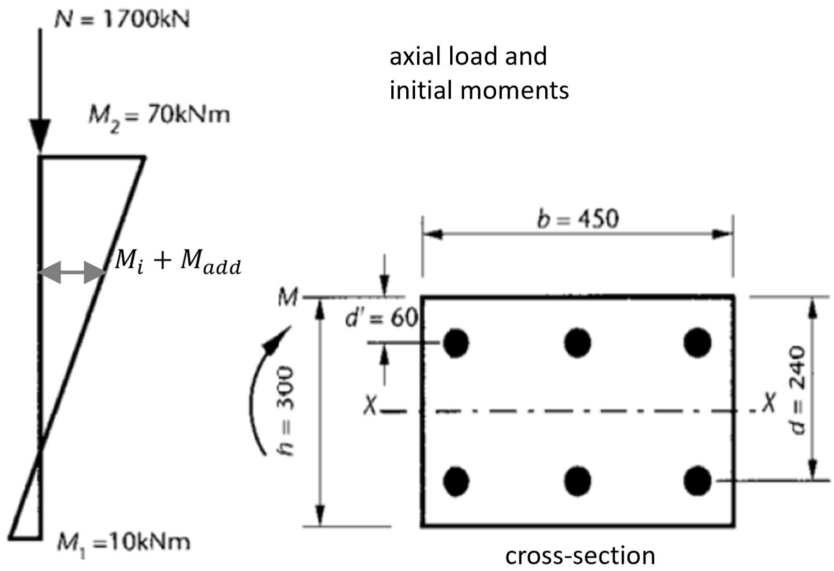

Figure 8.21: Design example of a slender braced column

Solution

Slenderness ratios are:

$$\frac{l_{ex}}{h} = \frac{6.75}{0.3} = 22.5 > 15, \quad \frac{l_{ey}}{h} = \frac{8.0}{0.45} = 17.8 > 15 \tag{8.8}$$

Thus, the column is slender. As the column is bent in double curvature:

- $M_1 = -10\,\mathrm{kNm}$, $M_2 = 70\,\mathrm{kNm}$,
- $M_i = 0.4M_1 + 0.6M_2 = 0.4 \cdot (-10) + 0.6 \cdot 70 = 38\,\mathrm{kNm} > 0.4M_2 = 0.4 \cdot 70 = 28\,\mathrm{kNm}$,

so $M_i = 38\,\mathrm{kNm}$.

Taking an initial value $K_i = 1.0$, the additional moment is:

$$\beta_a = \frac{1}{2000}\left(\frac{l_e}{b}\right)^2, \quad a_u = \beta_a K h \tag{8.9}$$

$$M_{\text{add}} = N a_u = N\frac{1}{2000}\left(\frac{l_e}{b}\right)^2 Kh = 1700 \cdot \frac{1}{2000}\left(\frac{6750}{300}\right)^2 \cdot 1 \cdot 300 \cdot 10^{-3} = 129\,\mathrm{kNm} \tag{8.10}$$

1st Iteration

Trial moment:

$$M_t = M_i + M_{\text{add}} = 38 + 129 = 167\,\mathrm{kNm} \tag{8.11}$$

$$\frac{N}{bhf_{cu}} = \frac{1700 \cdot 10^3}{450 \cdot 300 \cdot 30} = 0.420, \quad \frac{M}{bh^2 f_{cu}} = \frac{167 \cdot 10^6}{300 \cdot 300^2 \cdot 30} = 0.137 \tag{8.12}$$

From the design chart with $d/h = 0.8$:

- $\frac{A_{sc}}{bh}\frac{f_y}{f_{cu}} = 0.50 \implies \frac{A_{sc}}{bh} = 3\%$.

With $100A_{sc}/bh = 3$, determine K using:

$$K = \frac{N_{uz} - N}{N_{uz} - N_{\text{bal}}} = 0.67 \leq 1 \tag{8.13}$$

This new $K = 0.67$ is used to recalculate M_{add} and M_t for the second iteration.

2nd Iteration

The trial moment $M_t = M_i + M_{\text{add}}$ is used with the design chart to determine $(100A_{sc})/bh$. Iterations continue until K values converge, which occurs after two iterations.

Table 8.4: Iteration Table for Slender Column Design

iter	K	M_t	$M_t/(bh^2)$	$100A_{sc}/(bh)$	K
(1)	1	167	4.12	3.0	0.67
(2)	0.67	124	3.06	2.1	0.60

The steel area required is:

$$A_{sc} = \frac{2.1bh}{100} = \frac{2.1(450 \cdot 300)}{100} = 2835\,\mathrm{mm}^2 \quad (\rho = 2.1\%) \tag{8.14}$$

As a check:

$$N_{\text{bal}} = 0.25f_{cu}bd = 0.25 \cdot 30 \cdot 450 \cdot 240 \cdot 10^{-3} = 810\,\mathrm{kN} \tag{8.15}$$

$$N_{uz} = 0.45f_{cu}bd + 0.87f_yA_{sc} = (0.45 \cdot 30 \cdot 450 \cdot 300 + 0.87 \cdot 500 \cdot 2835) \cdot 10^{-3} = 3055.7\,\mathrm{kN} \tag{8.16}$$

$$K = \frac{N_{uz} - N}{N_{uz} - N_{\text{bal}}} = \frac{3055.7 - 1700}{3055.7 - 810} = 0.604 \tag{8.17}$$

This agrees with the final $K = 0.60$ from the design chart.

8.5.6 Unbraced Slender Columns

The sway of an unbraced structure causes larger additional moments in the columns. The distribution of moments is shown in [HKCC2013: Figure 6.17]. The additional moment M_{add} may be assumed to occur at the end with the stiffer joint. Thus, the maximum design moment M_d always occurs at the stiffer end joint. The design procedure follows that for braced slender columns once M_d is determined.

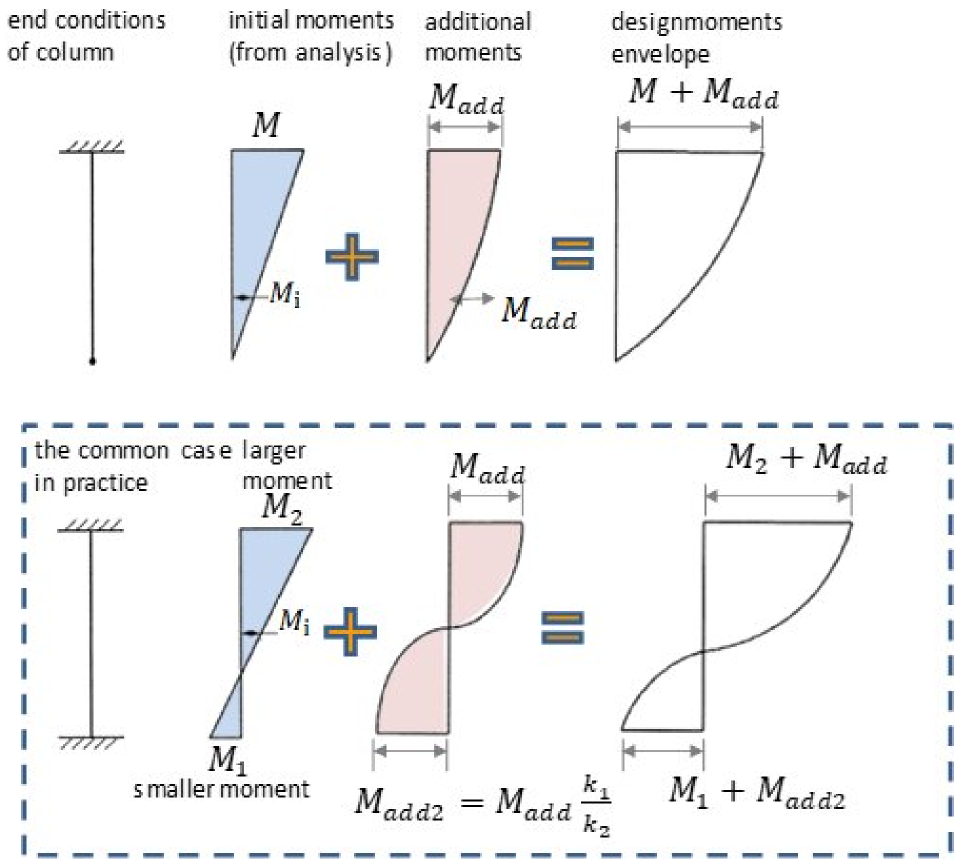

Figure 8.22: Design moments of unbraced slender columns

Chapter 9

Design Examples

9.1 A continuous beam

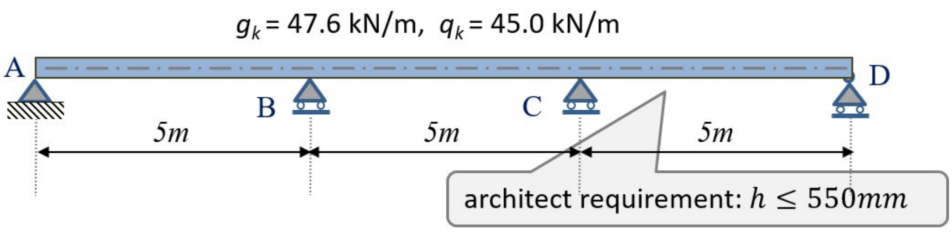

Figure 9.1: Continuous 3-span beam.

- We will design the beam for flexure, shear, and calculate the anchorage lengths.
- Typically we don't know the dimensions of the cross-section but we need to estimate them (preliminary design).
- Assume there is an architectural requirement: $h \leq 550\,\mathrm{mm}$.

Solution breakdown

- Structural analysis (subsection 9.1.1).
- Flexural design (subsection 9.1.2).
- Shear design (subsection 9.1.3).
- Anchorage design (subsection 9.1.4).
- Detailing Issues (subsection 9.1.5).

9.1.1 Structural analysis

Firstly, we need to define all necessary load combinations. To determine the maximum (in absolute value) bending moments and shear forces of the examined beam, we need three load combinations taking into account the symmetry of the structure. All structural analyses were performed with the aid of the online solver "https://structural-analyser.com/".

Load Combination 1 (Figure 9.3):

- $1.4G_k + 1.6Q_k$ on spans AB and CD, $1.0G_k$ on span BC.
- Captures M_{max} at spans AB and CD (maximum positive moment in the outer spans) and maximum shear forces and reaction forces at supports A and D
- Captures also M_{min} at span BC in case that moment is a negative (hogging) moment.

Load Combination 2 (Figure 9.4):

- $1.0G_k$ on spans AB and CD, $1.4G_k + 1.6Q_k$ on span BC.
- Captures M_{max} at span BC (maximum positive moment in the middle span).

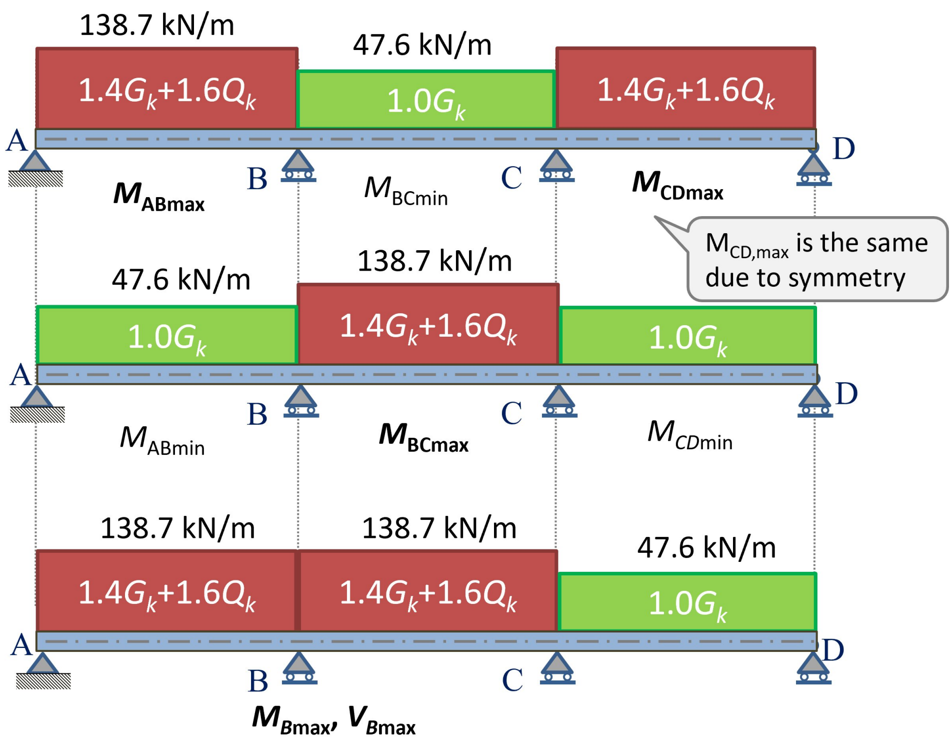

Figure 9.2: 3 load combinations are needed for this structure.

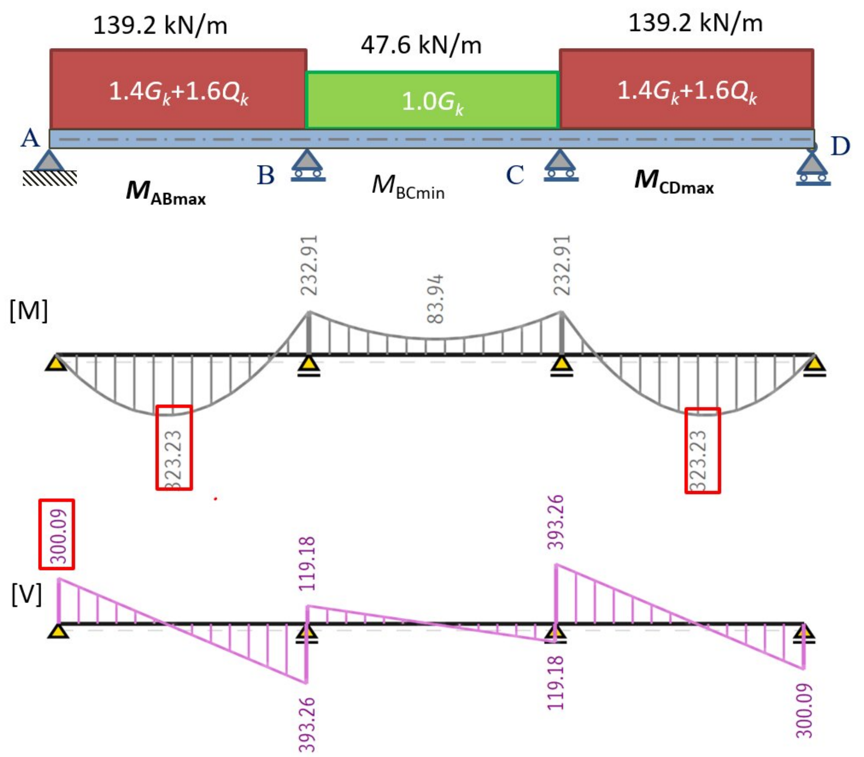

Figure 9.3: Mmax at spans AB, CD load combination: Bending moments and shear forces diagrams.

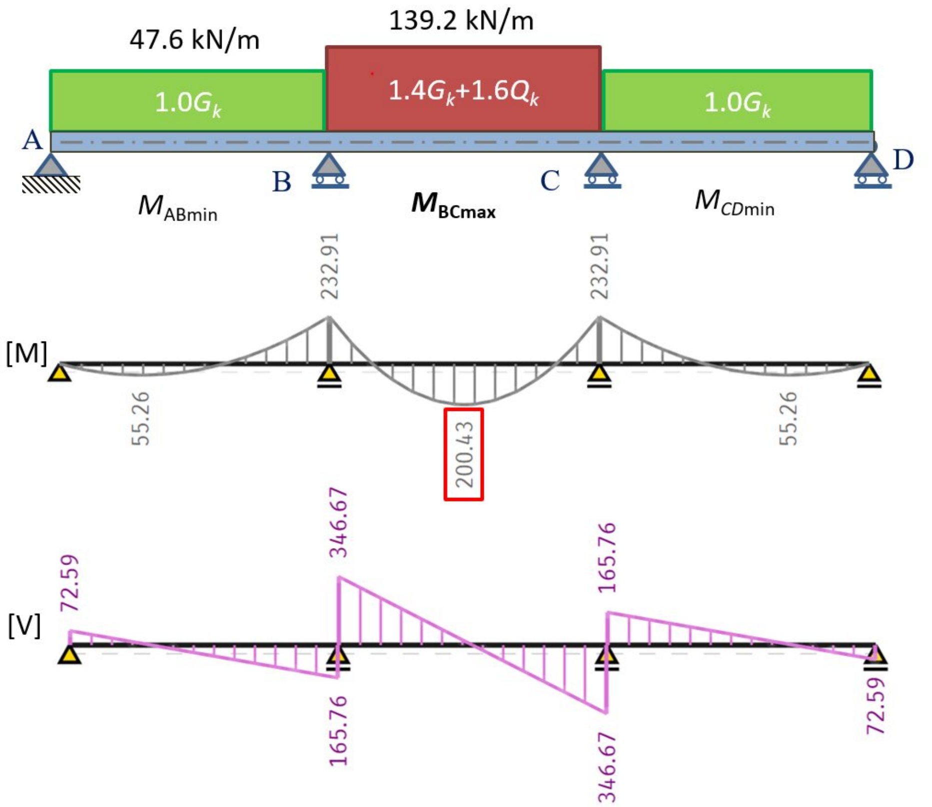

Figure 9.4: Mmax at span BD load combination: Bending moments and shear forces diagrams.

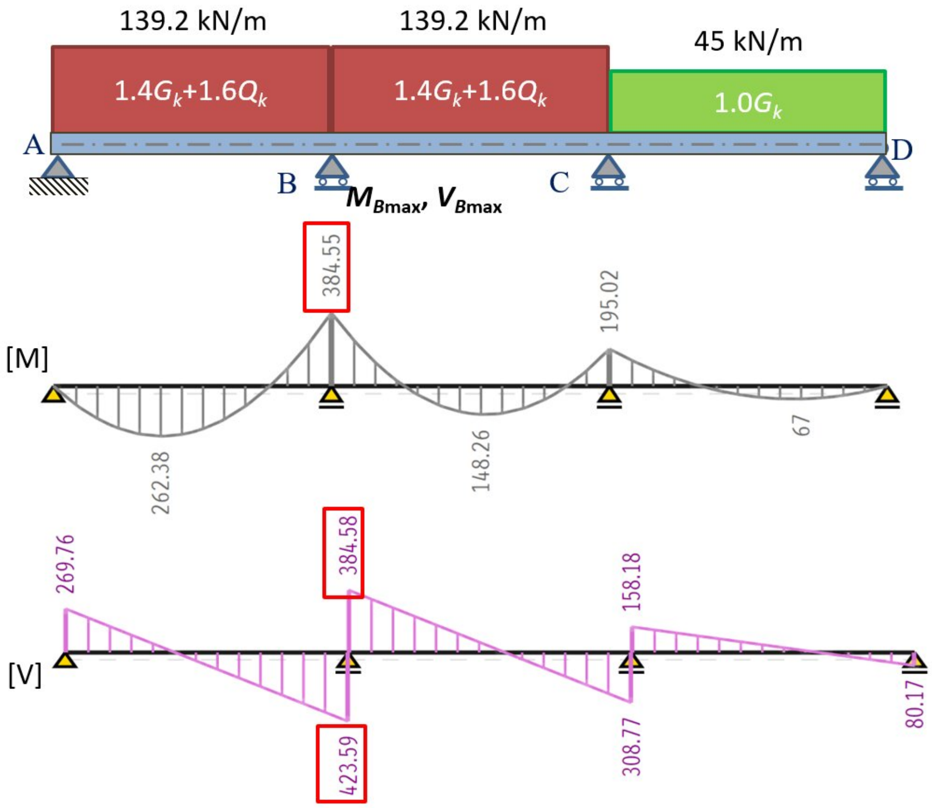

Figure 9.5: Mmax at B and C load combination: Bending moments and shear forces diagrams.

Load Combination 3 (Figure 9.5):

- $1.4G_k + 1.6Q_k$ on spans AB and BC, $1.0G_k$ on span CD.
- Captures M_{max} at support B (maximum negative moment at support B) and because of symmetry at support C, and maximum shear forces and reaction forces at support B (and because of symmetry also at support C).

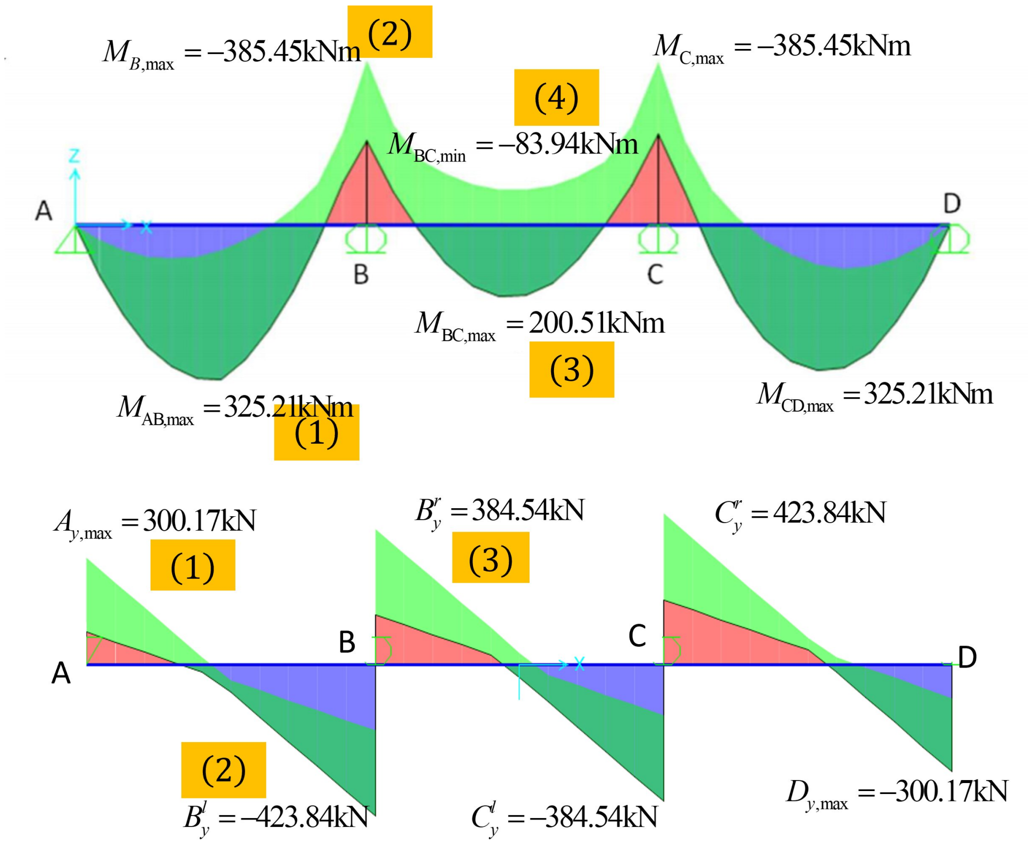

Figure 9.6: Top: bending moments envelope. Bottom: shear forces envelope.

Bending Moments and Shear Forces Envelope

Figure 9.6 illustrates the bending moment envelope of the examined 3-span continuous beam with equal spans. The bending envelope provides a comprehensive view of the maximum moments across the beam, derived from the critical load combinations in Figure 9.3, Figure 9.4, and Figure 9.5. Specifically, the envelope of Figure 9.6 identifies four critical moments, essential for designing the beam:

1. M_{max} at Spans AB, CD (Maximum Positive Moment in Outer Spans):
   - *Location*: Near midspan of spans AB and CD.
   - *Value*: Determined from LC1 (Figure 9.3).
   - *Symmetry*: The moments in spans AB and CD are equal due to the symmetry of the beam and the load combination.
2. M_{max} at Span BC (Maximum Positive Moment in Middle Span):
   - *Location*: Midspan of span BC.

   - *Value*: Determined from LC2 (Figure 9.4).

3. M_{max} at Support B (Maximum Negative Moment at Support B):

   - *Location*: At support B.
   - *Value*: Determined from LC3 (Figure 9.5).

4. M_{max} at Support C (Maximum Negative Moment at Support C equal because of symmetry with the Maximum Negative Moment at Support B),

5. The envelope (Figure 9.6) also accounts for symmetry by flipping the bending moment diagram for support B to capture the maximum negative moment at support C.

A similar approach is used for the shear force envelope, also shown in Figure 9.6, which identifies maximum shear forces for shear reinforcement design. In summary, the bending moments and shear forces envelopes summarize the worst-case scenarios (ULS) in all spans and supports for which the beam should be designed, making it a fundamental tool for structural design.

9.1.2 Flexural design

The next step is to identify the design cases that we need to solve for.

Design Cases:

1. *Moment in span AB*: Positive (sagging) moment in the span between A and B. Because of symmetry, the same reinforcement requirements will determine the reinforcement of span CD .

2. *Moment in span $BC+$*: Positive (sagging) moment (M_{BC+}) in the span BC.

3. *Moment in span $BC-$*: Negative (hogging) moment (M_{BC-}) in the span BC, requiring reinforcement at the top.

4. *Moment at support B*: Negative (hogging) moment at support B. Because of symmetry, the same reinforcement requirements will determine the reinforcement of support C.

The design should start from the most demanding case, which since the cross-section is rectangular (and hence symmetric to positive and negative moments) it is the case with the highest bending moment demand: Moment at support B.

Preliminary design

Assume we would like to design a singly reinforced beam and not consider moment redistribution. We can estimate the dimensions of the cross-section setting the maximum value of K that appears in the beam as equal to $K' = 0.156$:

$$K_{max} = \frac{M_{b,max}}{bd^2 f_{cu}} \leq K' = 0.156 \rightarrow$$
$$bd^2 \geq \frac{M_{b,max}}{0.156 f_{cu}} = \frac{385.5\,\mathrm{Nmm} \cdot 10^6}{0.156 \cdot 30\,\mathrm{N/mm^2}} = 0.0825\,\mathrm{m}^3$$

- Choosing $b = 300\,\mathrm{mm}$ gives $\rightarrow$d = $524\,\mathrm{mm}$

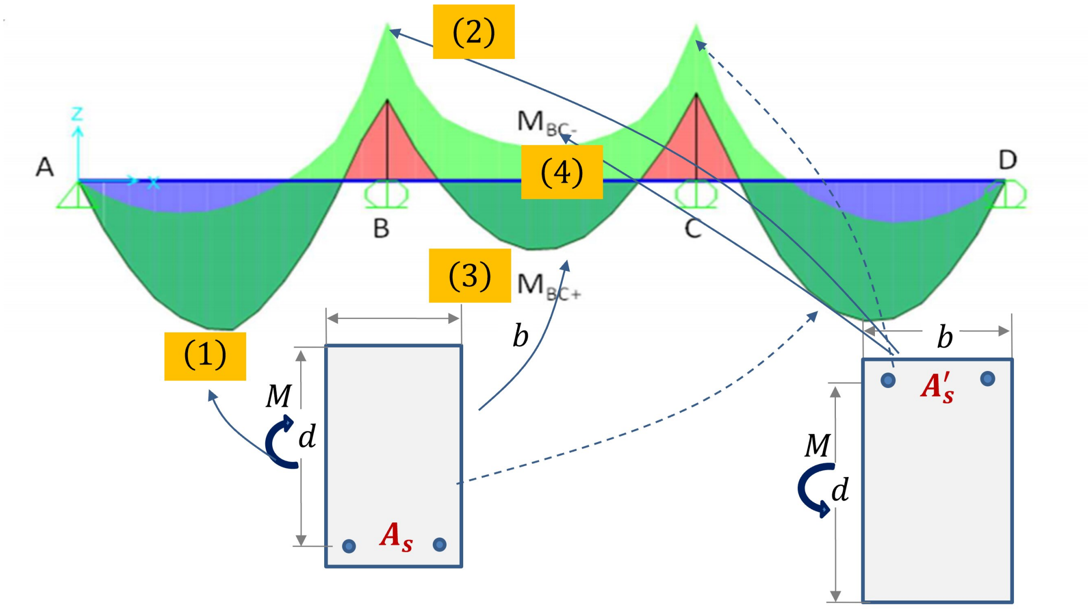

Figure 9.7: Design cases for flexure.

- Thus, we would like to choose $d = 550\,\text{mm}$ but to comply with the architect requirement: $h \leq 550\,\text{mm}$, **we choose** $d = 500\,\text{mm}$ and $b = 300\,\text{mm}$ ($h = 550\,\text{mm}$)
- Note: We estimate the self-weight of the beam which is part of the dead load ($g_k = 47.6\,\text{kN/m}$). If the dimensions of the cross-section have to be later revised we might need to update the dead load and redo the structural analysis.

support B: $M_{B,\textbf{max}} = -385.45\,\text{kN m}$ (see Figure 9.6)

- Dimensionless bending moment ratio K:

$$K = \frac{385.5\,\text{kN m}}{(0.30\,\text{m})(0.50\,\text{m})^2 30\,\text{N/mm}^2} = 0.171 > K' = 0.156$$

  We have to design a **doubly** R/C section (provide compression steel).

- Compression steel area required A'_s (Equation 2.14):

$$A'_s = \frac{(0.172 - 0.156) \cdot 30\,\text{N/mm}^2 \cdot 300\,\text{mm} \cdot (500\,\text{mm})^2}{0.87 \cdot 500\,\text{N/mm}^2 \cdot (500\,\text{mm} - 50\,\text{mm})} \rightarrow A'_s = 179\,\text{mm}^2$$

- Tension steel area required A_s (Equation 2.15):

$$A_s = \frac{0.156 \cdot 30 \cdot 300 \cdot (500)^2}{0.87 \cdot 500 \cdot (0.775 \cdot 500)} + 179 \rightarrow A_s = 2261\,\text{mm}^2$$

- Has compression steel yielded? Yes, because:

$$\frac{d'}{x} = \frac{50\,\text{mm}}{0.5 \cdot 500\,\text{mm}} = 0.20 \leq 0.38$$

- Provide:Tension steel **5T25** $A_s = 2454\,\text{mm}^2$
- Provide: Compression steel **2T16** $A_s = 402\,\text{mm}^2$ (see Figure 9.10)

- Check the **steel ratio**:

$$\rho_{min} = 0.3\% < \rho = \frac{A_s}{b \cdot h} = \frac{2454}{300 \cdot 550} = 1.5\% < \rho_{max} = 2.5\% \rightarrow \text{OK.}$$

AB mid-span $M_{AB,\mathbf{max}} = 325.21\,\mathrm{kN\,m}$ (see Figure 9.6)

- Dimensionless bending moment ratio K:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{325.2\,\mathrm{kN\,m}}{(0.30\,\mathrm{m})(0.5\,\mathrm{m})^2 30\,\mathrm{N/mm^2}} = 0.145 < K' = 0.156$$

We have to design a **singly** reinforced section.

- Lever arm z (check that $z_{\text{min}} = 0.775d < z < z_{\text{max}} = 0.95d$):

$$z = 500 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.145}{0.9}}\right) = 400\,\mathrm{mm}$$

- Tension steel area required A_s (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{325.2\,\mathrm{kN\,m} \cdot 10^6}{0.87 \cdot 500 \cdot 400} = 1871\,\mathrm{mm^2}$$

- Provide **4T25**: $A_s = 1963\,\mathrm{mm^2}$ (see Figure 9.10)
- Check the **steel ratio**:

$$\rho_{min} = 0.3\% < \rho = \frac{A_s}{b \cdot h} = \frac{1963}{300 \cdot 550} = 1.2\% < \rho_{max} = 2.5\% \rightarrow \text{OK.}$$

BC mid-span: $M_{BC,\mathbf{max}} = 200.51\,\mathrm{kN\,m}$ (see Figure 9.6)

- Dimensionless bending moment ratio K:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{200.5\,\mathrm{kN\,m}}{(0.30\,\mathrm{m}) \cdot (0.5\,\mathrm{m})^2 \cdot 30\,\mathrm{N/mm^2}} = 0.089 < K' = 0.156$$

We have to design a **singly** reinforced section.

- lever arm z (check that $z_{\text{min}} = 0.775d < z < z_{\text{max}} = 0.95d$):

$$z = 500 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.089}{0.9}}\right) = 444\,\mathrm{mm}$$

- Tension steel area required A_s (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{200.5\,\mathrm{kN\,m} \cdot 10^6}{0.87 \cdot 500 \cdot 444} = 1037\,\mathrm{mm^2}$$

Provide **2T20+1T25**: $A_s = 1119\,\mathrm{mm^2}$ (see Figure 9.10)

- Check the **steel ratio**:

$$\rho_{min} = 0.3\% < \rho = \frac{A_s}{b \cdot h} = \frac{1119}{300 \cdot 550} = 0.68\% < \rho_{max} = 2.5\% \rightarrow \text{OK.}$$

BC mid-span: $M_{BC,\mathbf{max}} = -83.94\,\mathrm{kN\,m}$ (see Figure 9.6)

- Dimensionless bending moment ratio K:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{83.94\,\mathrm{kN\,m}}{(0.30\,\mathrm{m})(0.5\,\mathrm{m})^2 30\,\mathrm{N/mm^2}} = 0.037 < K' = 0.156$$

  We have to design a **singly** reinforced section.

- Calculate the lever arm z:

$$z = 500 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.037}{0.9}}\right) = 478\,\mathrm{mm}$$

  Check that $z = 478\,\mathrm{mm} > z_{\max} = 0.95d = 475\,\mathrm{mm} \rightarrow$ set$z = z_{\max} = 0.95d = 475\,\mathrm{mm}$

- Tension steel area required A_s (Eq 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{83.94\,\mathrm{kN\,m} \cdot 10^6}{0.87 \cdot 500 \cdot 475} = 406\,\mathrm{mm^2}$$

- Provide **2T18**: $A_s = 509\,\mathrm{mm^2}$ (see Figure 9.10)

- Check the **steel ratio**:

$$\rho_{min} = 0.3\% < \rho = \frac{A_s}{b \cdot h} = \frac{509}{300 \cdot 550} = 0.31\% < \rho_{max} = 2.5\% \rightarrow \text{OK.}$$

9.1.3 Shear design

- **Note:** we don't have the dimensions of the supports/columns x_1, x_2 (see Fig 9.8) yet to calculate the shear forces at the face of the supports V_s ($V_s = V_c - q \cdot \frac{x}{2}$) we could either suppose them or ignore them

- here we will ignore them $\rightarrow$ it is more conservative as the shear forces V_c are higher

First we check if the design is admissible (ie if the section dimensions are sufficient). The maximum shear stress is:

$$v_{\max} = \frac{V}{b_v d} = \frac{423.84 \cdot 10^3}{300 \cdot 500} = 2.83\,\mathrm{N/mm^2} < \min\left\{0.8\sqrt{f_{cu}}, 7\,\mathrm{N/mm^2}\right\} = \min\{4, 7\} = 4\,\mathrm{N/mm^2}$$

Support A

- **design concrete shear stress v_c for 2T25 ($A_s = 982\,\mathrm{mm^2}$):**

$$v_c = 0.79 \left(\frac{f_{cu}}{25}\right)^{1/3} \left(\frac{100 A_s}{b_v d}\right)^{1/3} \left(\frac{400}{d}\right)^{1/4} \frac{1}{\gamma_m}$$

$$v_c = 0.79 \left(\frac{30}{25}\right)^{1/3} \left(\frac{100 \cdot 982}{300 \cdot 500}\right)^{1/3} \left(\frac{400}{\cancel{500}400}\right)^{1/4} \frac{1}{1.25} = 0.58\,\mathrm{N/mm^2}$$

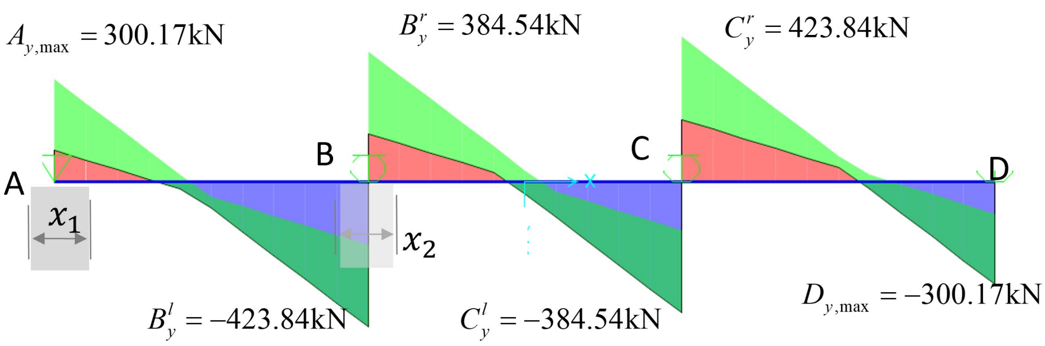

Figure 9.8: Beam with flexureal and shear reinforcement.

- **shear links (stirrups)**
  The shear force at the nominal section (at a distance d from support ~~face~~ center) is (Equation A.2)

$$V_d = V_q - qd = 300.2 - 138.7 \cdot 0.5 = 230.8\,\mathrm{kN}$$

  The corresponding shear stress at the critical section

$$v = \frac{V_d}{b_v d} = \frac{230.8 \cdot 10^3}{300 \cdot 500} = 1.54\,\mathrm{N/mm^2} > v_c + 0.4 = 0.98\,\mathrm{N/mm^2}$$

  Shear reinforcement is required $\rightarrow$ suppose mild steel (R) is used:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c) b_v}{0.87 f_{yv}} = \frac{0.96 \cdot 300}{0.87 \cdot 250} = 1.32$$

  For R10 links ($A_{sv} = 157\,\mathrm{mm^2}$) : $s_v = \frac{s_v}{A_{sv}} \cdot A_{sv} = 119\,\mathrm{mm} < 0.75d = 375\,\mathrm{mm}$

- Provide R10@110 $\left(\frac{A_{sv}}{s_v} = 1.43\right)$ see Figure 9.9

Support B^{left}

- **Design concrete shear stress** v_c **for 5T25 (**$A_s = 2454\,\mathrm{mm^2}$**):**

$$v_c = 0.79 \left(\frac{f_{cu}}{25}\right)^{1/3} \left(\frac{100 A_s}{b_v d}\right)^{1/3} \left(\frac{400}{d}\right)^{1/4} \frac{1}{\gamma_m} \rightarrow$$

$$v_c = 0.79 \left(\frac{30}{25}\right)^{1/3} \left(\frac{100 \cdot 2454}{300 \cdot 500}\right)^{1/3} \left(\frac{400}{\not{500}400}\right)^{1/4} \frac{1}{1.25} = 0.79\,\mathrm{N/mm^2}$$

- Longitudinal Steel Selection: Use tension steel at the location of the shear check based on the bending moment sign at the same location (according to the bending moment envelope); if uncertain, opt for the conservative minimum (e.g., 2 bars of 25mm).

- **shear links (stirrups)**
  The shear force at the nominal section (at a distance d from support ~~face~~ center) is (Eq A.2)

$$V_d = V_q - qd = 423.8 - 138.7 \cdot 0.5 = 354.5\,\mathrm{kN}$$

  The corresponding shear stress at the critical section

$$v = \frac{V_d}{b_v d} = \frac{354.5 \cdot 10^3}{300 \cdot 500} = 2.36\,\mathrm{N/mm^2} > v_c + 0.4 = 1.19\,\mathrm{N/mm^2}$$

Shear reinforcement is required $\rightarrow$ suppose mild steel (R) is used:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_v}{0.87f_{yv}} = \frac{1.57 \cdot 300}{0.87 \cdot 250} = 2.17$$

For R12 links ($A_{sv} = 226\,\mathrm{mm}^2$) : $s_v = \frac{s_v}{A_{sv}} \cdot A_{sv} = 104\,\mathrm{mm} < 0.75d = 375\,\mathrm{mm}$

- Provide R12@100 $\left(\frac{A_{sv}}{s_v} = 2.26\mathrm{mm}\right)$ see Figure 9.9

Support B^{right}

- Design concrete shear stress v_c for 5T25 ($A_s = 2454\,\mathrm{mm}^2$) : $v_c = 0.79\,\mathrm{N/mm}^2$
- **shear links (stirrups)**
  The shear force at the nominal section (at a distance d from support ~~face~~ center) is (Equation A.2)

$$V_d = V_q - qd = 384.5 - 138.7 \cdot 0.5 = 315.2\,\mathrm{kN}$$

The corresponding shear stress at the critical section

$$v = \frac{V_d}{b_v d} = \frac{315.2 \cdot 10^3}{300 \cdot 500} = 2.10\,\mathrm{N/mm}^2 > v_c + 0.4 = 1.19\,\mathrm{N/mm}^2$$

Shear reinforcement is required $\rightarrow$ suppose mild steel (R) is used:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_v}{0.87f_{yv}} = \frac{1.31 \cdot 300}{0.87 \cdot 250} = 1.81$$

For R12 links ($A_{sv} = 226\,\mathrm{mm}^2$) : $s_v = \frac{s_v}{A_{sv}} \cdot A_{sv} = 124\,\mathrm{mm} < 0.75d = 375\,\mathrm{mm}$

- Provide R12@120 $\left(\frac{A_{sv}}{s_v} = 1.88\mathrm{mm}\right)$ see Figure 9.9

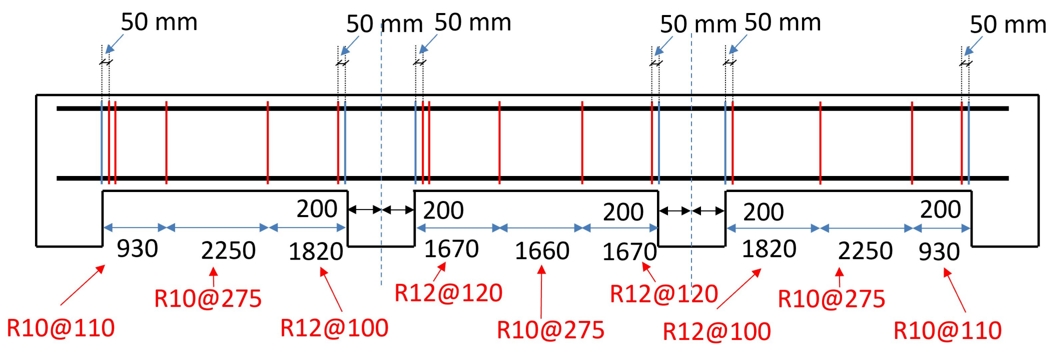

Figure 9.9: Beam with flexureal and shear reinforcement.

span AB

- the minimum amount of required links (stirrups) is:

$$\left(\frac{A_{sv}}{s_v}\right)_{\min} = \frac{0.4\,b_v}{0.87f_{yv}} = \frac{0.4 \cdot 300}{0.87 \cdot 250} = 0.552\,\mathrm{mm} \rightarrow \text{provide R10@275:} \frac{A_{sv}}{s_v} = 0.571$$

- shear resistance of concrete plus minimum links (v_c for 4T25)

$$V_n = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right) 0.87 f_{yv} d$$
$$V_n = (0.73 \cdot 300 \cdot 500 + 0.571 \cdot 0.87 \cdot 250 \cdot 500) \cdot 10^{-3} = 172.0\,\mathrm{kN}$$

- Identify the location where the shear force drops to the level of the shear resistance of concrete plus minimum links using Equation A.2:

$$\frac{V_n - V_q}{q} = \frac{300.2 - 172.0}{138.7} \cdot 1000 \cdot x_1 = 924\,\mathrm{mm}$$
$$\frac{V_n - V_q}{q} = \frac{423.8 - 172.0}{138.7} \cdot 1000 \cdot x_2 = 1815\,\mathrm{mm}$$
$$\Delta x = 5\,\mathrm{m} - x_1 - x_2 = 2250\,\mathrm{mm}$$

- in the middle $2250mm$ of the beam we provide the minimum links $R10@275$ (Figure 9.9)

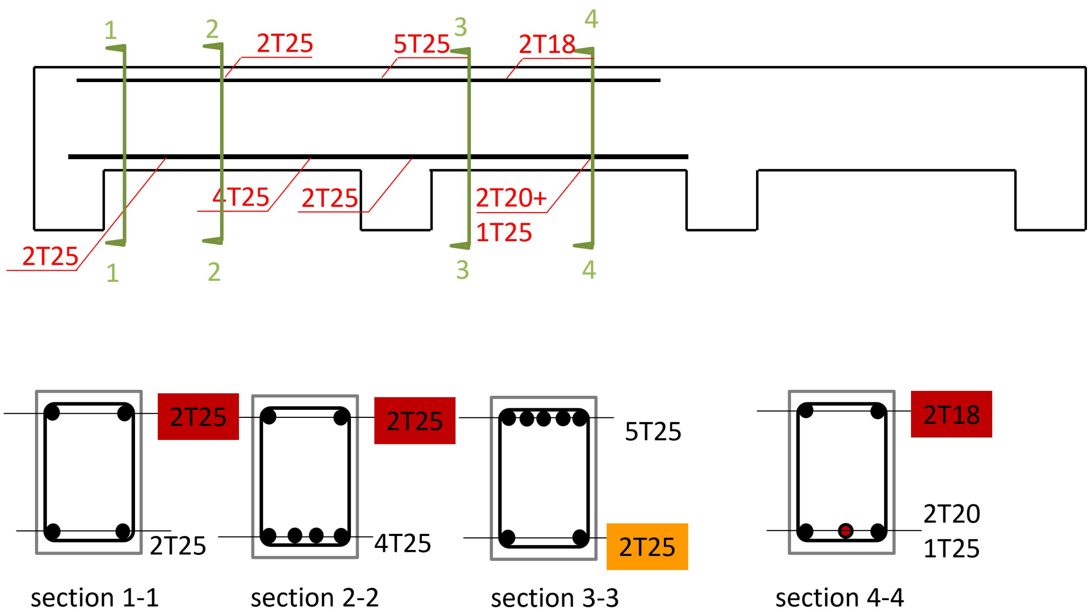

Figure 9.10: Longitudinal section and cross-sections.

span BC

- the minimum amount of required links (stirrups) is: provide $R10@275$: $\frac{A_{sv}}{s_v} = 0.571$
- shear resistance of concrete plus minimum links (v_c for 2T20 + 1T25)

$$V_n = v_c b_v d + \left(\frac{A_{sv}}{S_v}\right) 0.87 f_{yv} d$$
$$V_n = (0.61 \cdot 300 \cdot 500 + 0.571 \cdot 0.87 \cdot 250 \cdot 500) \cdot 10^{-3} = 153.5\,\mathrm{kN}$$

- envelope of the shear resistance of concrete plus minimum links

$$\frac{V_n - V_q}{q} = \frac{384.5 - 153.5}{138.7} \cdot 1000 \cdot x = 1665\,\mathrm{mm}$$
$$\Delta x = 5\,\mathrm{m} - 2x = 1660\,\mathrm{mm}$$

- in the middle $1660mm$ of the beam we provide the minimum links $R10@275$ (Figure 9.9)

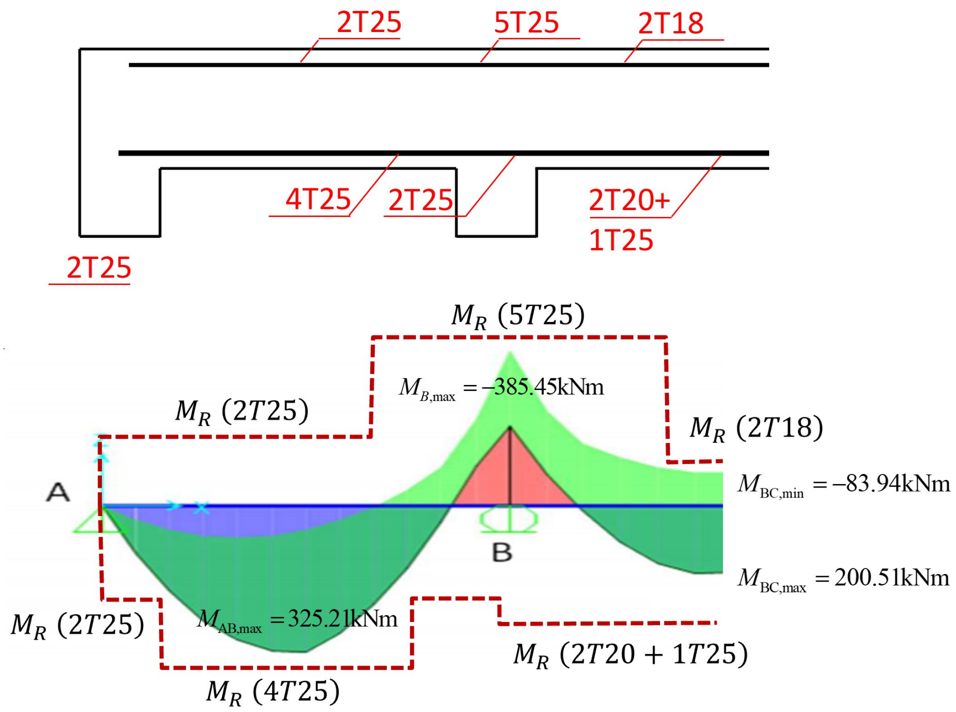

Figure 9.11: Envelope of moment demand and of moment capacity.

9.1.4 Anchorage design

We calculate the anchorage for **straight** bars.

- Calculate the anchorage length in tension for T18, T20, T25 from Table 8.4 (HKCC2013 [7]): $K_A = 40$

$$l_b = K_A \phi = 40 \cdot 25 = 1000\,\text{mm}$$
$$l_b = K_A \phi = 40 \cdot 20 = 800\,\text{mm}$$
$$l_b = K_A \phi = 40 \cdot 18 = 720\,\text{mm}$$

- The total anchorage length required is

$$l_b + \frac{d}{2} = 1000\,\text{mm} + 250 = 1250\,\text{mm}$$
$$l_b + \frac{d}{2} = 800\,\text{mm} + 250 = 1050\,\text{mm}$$
$$l_b + \frac{d}{2} = 720\,\text{mm} + 250 = 970\,\text{mm}$$

If the anchorage was **bend** by 90°

- the effective anchorage length of a standard 90° bend would be $l_e = 12\phi$

$$l_e = 1350\,\text{mm} - 12 \cdot 25 = 950\,\text{mm}$$
$$l_e = 1350\,\text{mm} - 12 \cdot 20 = 810\,\text{mm}$$

- The straight part of the bar is $(40\phi + d/2) - 12\phi$

$$(40\phi + \frac{d}{2}) - 12\phi = 1277\,\text{mm}$$

- The total anchorage length is equal to the straight length + $\pi(r/2 + \phi/4) + 4\phi$

$$= 950\,\text{mm} + \pi(r/2 + \phi/4) + 4\phi = 1227\,\text{mm}$$
$$= 810\,\text{mm} + \pi(r/2 + \phi/4) + 4\phi = 1031\,\text{mm}$$
$$= 754\,\text{mm} + \pi(r/2 + \phi/4) + 4\phi = 953\,\text{mm}$$

9.1.5 Detailing Issues

How many bars can we put in one layer?

We need to consider that:

- usually the maximum aggregate size = $20\,\text{mm}$
- practical clear distance in between neighboring bars should be around $25\,\text{mm}$ for proper concrete flow during concrete casting.
- assuming bar size = 25 and cover = $50\,\text{mm}$, Within a $300\,\text{mm}$ wide beam, subtract $50\,\text{mm}$ cover on each side until the centroid of the corner bards, leaves $200\,\text{mm}$ center to center distance between the corner bars, which means the free width is $200\,\text{mm}$ - 25= $175\,\text{mm}$.
- in the remaining $175\,\text{mm}$ width there is just enough space for 2T25 bars and the three 25mm gaps between them see Figure 9.12 (bottom right).
- So within a $300\,\text{mm}$ wide beam, we can barely fit five 25mm bars (5T25) snugly
- If one of these bars were of larger diameter (e.g., 32mm) we would not have enough space.

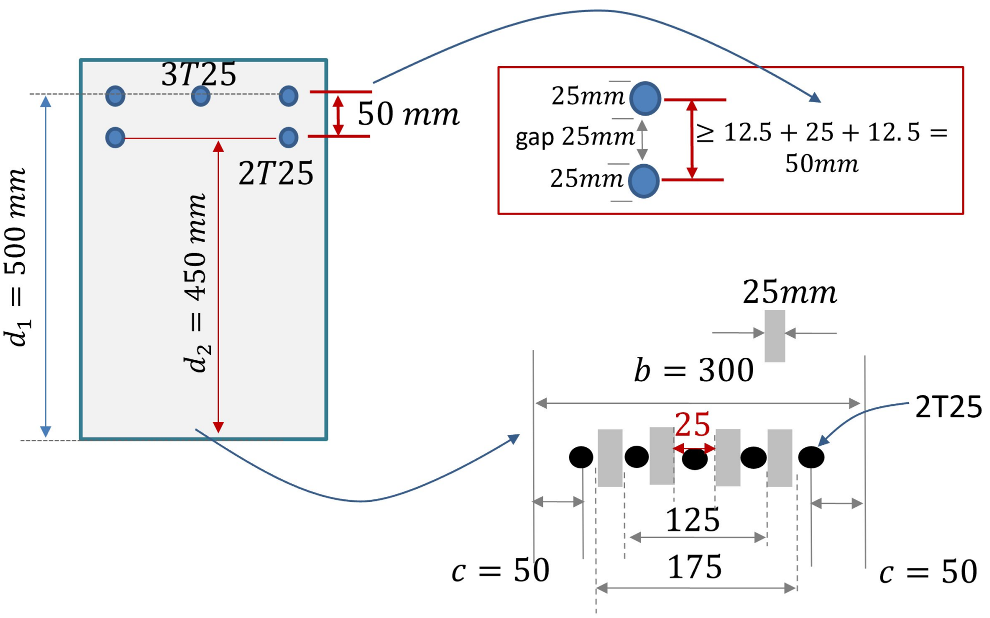

Figure 9.12: Detailing of the cross-section.

Detailing: Support B

If we couldn't fit the selected bar (e.g., 5T25) in one layer, we could arrange them in two layers (e.g., 3T25 + 2T25). What would change if we arrange the same reinforcement area in 2 layers e.g., (3T25 + 2T25) instead of 1 layer e.g., (5T25)?

Moment of resistance for 1 layer of A_s (5T25)

- moment equilibrium about FCC

$$M = 0.87 f_y A_s z + 0.87 f_y A_s' \left(\frac{z}{2} - d'\right)$$

- horizontal force equilibrium

$$0.45 f_{cu} b s + 0.87 f_y A_s' = 0.87 f_y A_s \rightarrow s = \frac{0.87 f_y (A_s - A_s')}{0.45 f_{cu} b}$$

In summary, arranging the total reinforcement area in multiple areas reduces the moment resistance due to a smaller effective depth (the lever arm for any additional layer decreases compared to the single layer, e.g., from 550mm to 450mm for the second layer).

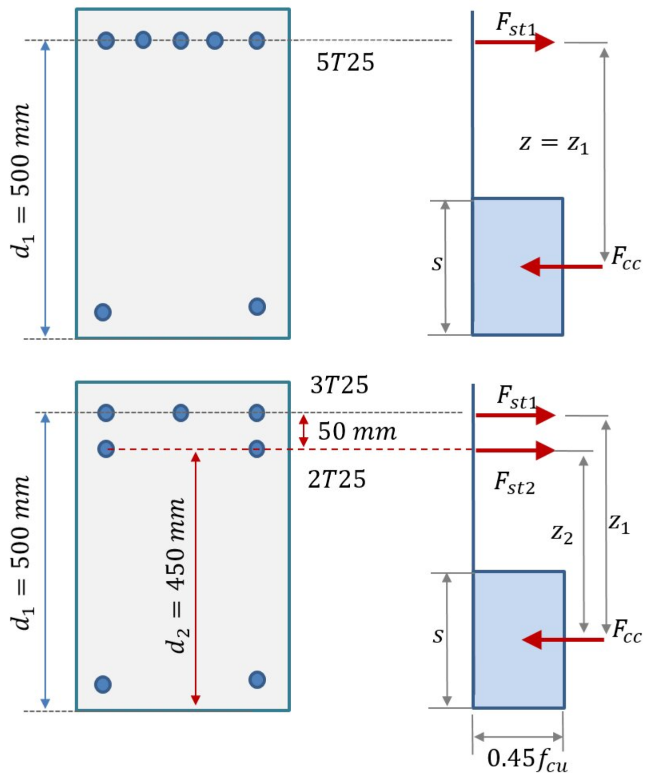

Figure 9.13: Arranging the same reinforcement in 2 vs 1 layer.

Moment of resistance for 2-layers of $A_s = A_{s1} + A_{s2}(3T25 + 2T25)$

- How will it affect the moment resistance?

- **horizontal force equilibrium**

$$F_{cc} = F_{st1} + F_{st2} \rightarrow 0.45 f_{cu} b s = 0.87 f_y A_{s1} + 0.87 f_y A_{s2} \rightarrow s = \frac{0.87 f_y (A_{s1} + A_{s2})}{0.45 f_{cu} b}$$

- s is the same as for 1-layer for the same total steel area: $A_{s1} + A_{s2} = A_s$
- **moment equilibrium about** F_{cc}

$$M_{R,\text{2layers}} = F_{st1} z_1 + F_{st2} z_2 = 0.87 f_y A_{s1} (d_1 - s/2) + 0.87 f_y A_{s2} (d_2 - s/2)$$
$$M_{R,\text{1layer}} = (F_{st1} + F_{st2}) z_1 = 0.87 f_y (A_{s1} + A_{s2}) z_1 = 0.87 f_y (A_{s1} + A_{s2})(d_1 - s/2)$$
$$M_{R,\text{1layer}} > M_{R,\text{2layers}}$$

9.2 Design a continuous R/C beam

This Section focuses on the design of reinforced concrete beams, building on concepts from previous Sections. The example discussed is a comprehensive recap with added complexity and details, particularly regarding curtailment of reinforcement bars and shear design.

Structural System

- Continuous beam with three equal spans of $5.2\,\mathrm{m}$.
- Flanged cross-section with web width $b_w = 300\,\mathrm{mm}$, overal depth $h = 600\,\mathrm{mm}$, and slab thickness (flange) $h_s = 460\,\mathrm{mm}$. Note the effective width of the flange b is not given, instead, we will estimate it.
- The beams are spaced at $4.0\,\mathrm{m}$ centers in the transverse direction, supporting a slab in one direction only (one-way slab).

Materials and Loads

- Material strengths $f_{cu} = 30\,\mathrm{N/mm^2}$ for concrete, $f_y = 500\,\mathrm{N/mm^2}$ for longitudinal steel, and $f_{yv} = 250\,\mathrm{N/mm^2}$ for links.
- Use a minimum **concrete cover** of $30\,\mathrm{mm}$ for mild exposure,
- **Dead load**: $g = 15\,\mathrm{kN/m^2}$ (self-weight of beam, slab, finishes, ceiling, and partitions).
- **Imposed load**: $q = 12.5\,\mathrm{kN/m^2}$. Both g and q are given in $\mathrm{kN/m^2}$ on the slab.
- **R/C specific weight**: $24.5\,\mathrm{kN/m^3}$ (per HK *Code of Practice for Dead and Imposed Loads 2011*).

Solution

9.2.1 Load Transfer and Design Load

- Slabs are one-way, transferring loads to beams. Beam spacing: $4\,\mathrm{m}$.
- Each beam carries load from 2 m on either side ($4\,\mathrm{m}$ tributary width).
- Dead load: $g_k = 15 \cdot 4 = 60\,\mathrm{kN/m}$,
- Imposed load: $q_k = 12.5 \cdot 4 = 50\,\mathrm{kN/m}$.
- Design load: $w = 1.4g_k + 1.6q_k = 1.4 \cdot 60 + 1.6 \cdot 50 = 164\,\mathrm{kN/m}$.
- Total design load per span: $F = 164 \cdot 5.2 = 852.8\,\mathrm{kN}$.

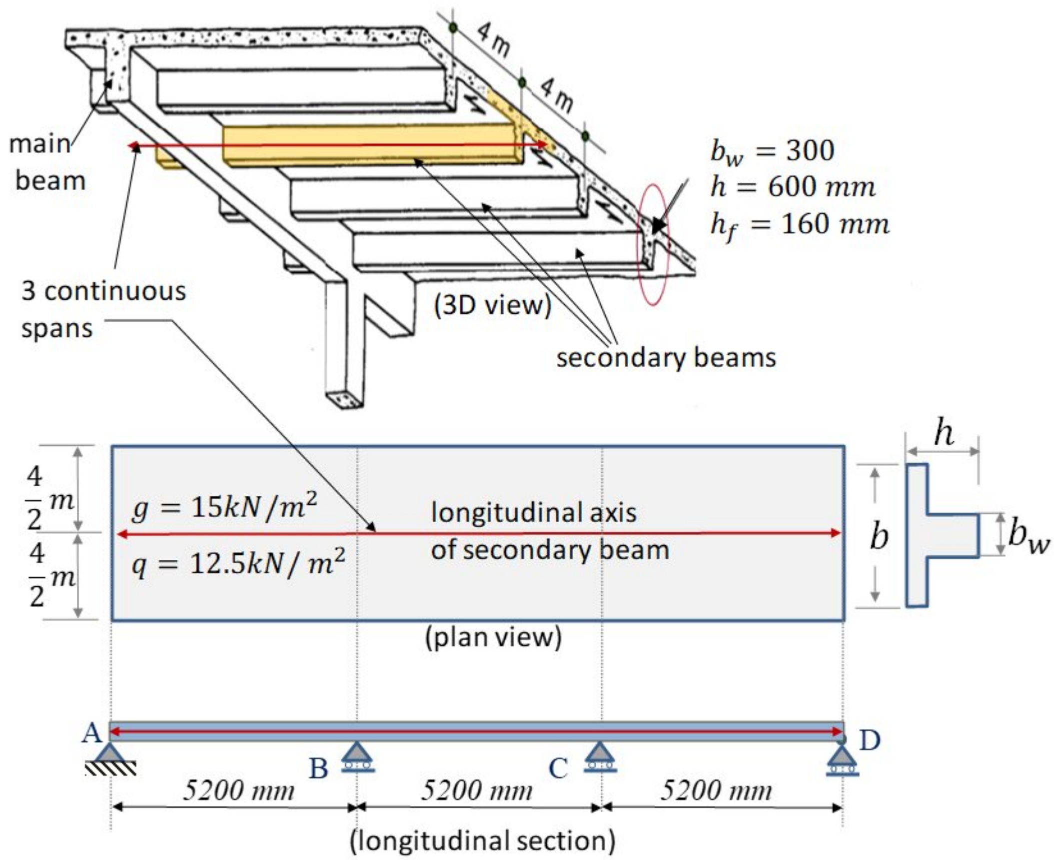

Figure 9.14: Structural system, flanged cross-section and three span beam.

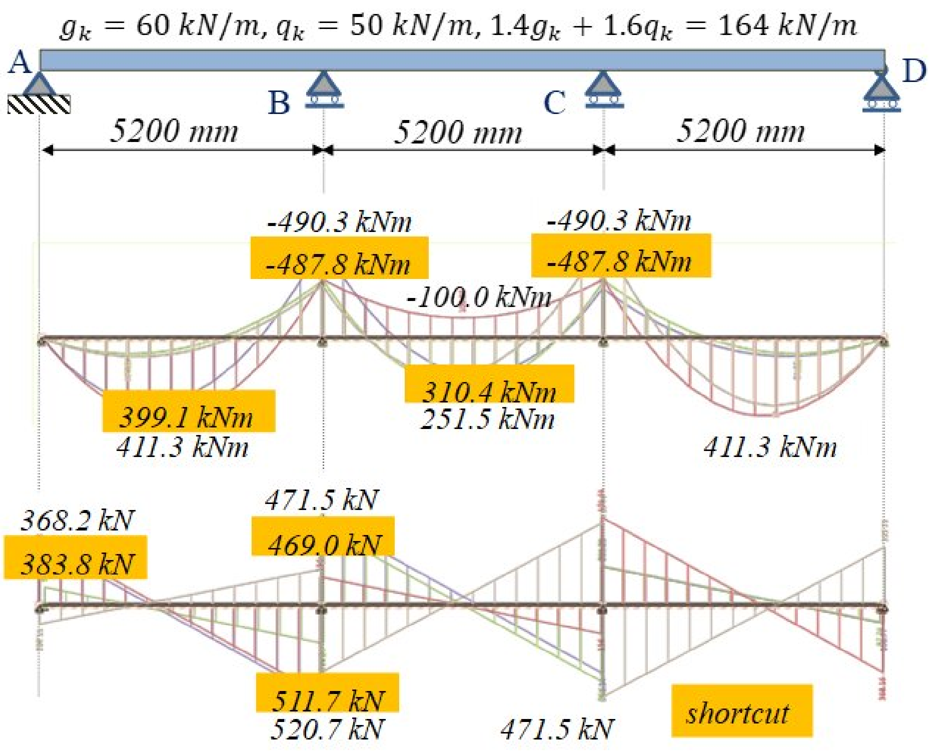

Figure 9.15: Bending moments envelope and shear forces envelope.

9.2.2 Structural Analysis

We keep the structural analysis description short, as it is essentially identical with the previous solved example (of Section 9.1.1). The bending moment diagram envelope is obtained by superimposing four load combinations, out of which we need to calculate three: M_{max} at spans AB and CD, M_{max} at span BC, and M_{max} at support B. The fourth load combination M_{max} at support C, is the same as M_{max} at support B, flipped because of symmetry. We similarly, construct the shear force envelope (see Figure 9.15).

The values highlighted with yellow color in Figure 9.15 are calculated with the shortcuts from HKCC2013, Clause 6.1.2.3, Table 6.1, since the design load is uniformly distributed, $q_k \leq g_k$, and spans are equal. Notice that while the code shortcuts provide satisfactory predictions of the actual values compared (e.g., 419 vs. 487 kN·m, 251 vs. 310 kN·m), they cannot calculate the negative midspan moment (e.g., -100 kN·m) of the BC midspan.

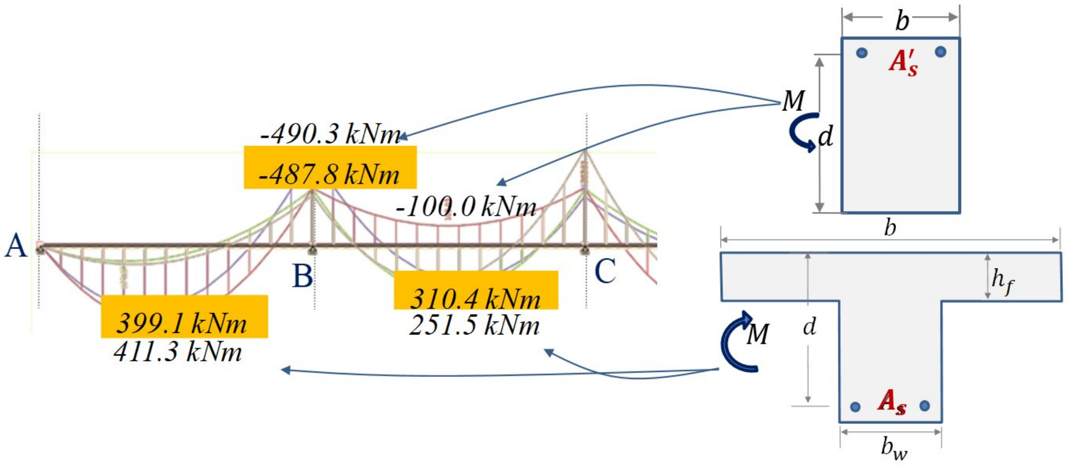

Figure 9.16: Four design cases for the flanged cross-section.

Design for Flexure at End Spans For midspans AB and CD, the **design moment is** $M_{AB} = 411.3$ **kNm**. For such a positive or sagging moment, the flange is under compression so the design of the section is that of a flanged T-section:

- **Effective depth**:

$$d = h - \left(30 + \phi_T + \frac{\phi_L}{2}\right) \approx 600 - (30 + 12 + 16) \approx 540\,\text{mm}$$

  where $\phi_T = 12\,\text{mm}$ (assumed link diameter) and $\phi_L = 32\,\text{mm}$ (assumed longitudinal bar diameter).

- **Effective flange width**:

$$b = b_w + \frac{L}{7} = 300 + \frac{5200}{7} = 1043\,\text{mm}$$

- **Check stress block depth**: The moment of resistance of the flange concrete is:

$$M_f = 0.45 f_{cu} b h_f \left(d - \frac{h_f}{2}\right) = 0.45 \cdot 30 \cdot 1043 \cdot 160 \cdot \left(540 - \frac{160}{2}\right) = 1036.3\,\text{kNm}$$

  Since $M_f > M_{AB} = 411.3\,\text{kNm}$, the stress block is within the flange ($s < h_f$).
  So the design the T-section as a **rectangular section** $b \times d$.

- **Dimensionless Moment**:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{411.3 \cdot 10^6}{1043 \cdot 540^2 \cdot 30} = 0.045 < K' \rightarrow$$

  Singly reinforced section.

- Lever arm:

$$z = 540 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.045}{0.9}}\right) = 511 mm < z_{\max} = 0.95d = 513\,\mathrm{mm}$$

- **Required tension steel area** (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{411.3 \cdot 10^6}{0.87 \cdot 500 \cdot 511} = 1849\,\mathrm{mm}^2$$

  Provide 4T25 ($A_s = 1963\,\mathrm{mm}^2$) bottom steel

- Check $\rho_{min} = 0.3\% < \rho = \frac{A_s}{b_w \cdot d} = \frac{1963}{300 \cdot 540} = 1.21\% < \rho_{max} = 2.5\% \rightarrow$ OK.

Design for Flexure at Interior Span (Positive Moment) For span BC, the design moment is $M_{BC} = 251.5\,\mathrm{kNm}$. Design as a flanged T-section, with the stress block within the flange ($s < h_f$) as confirmed previously:

- **Moment coefficient**:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{251.5 \cdot 10^6}{1043 \cdot 540^2 \cdot 30} = 0.028 < 0.156 = K'$$

  Since $K < K'$, no compression reinforcement is needed.

- Lever arm:

$$z = 540 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.028}{0.9}}\right) = 523\,\mathrm{mm}$$

  Check: $z > z_{\max} = 0.95d = 0.95 \cdot 540 = 513$ mm, so use $z = z_{\max} = 513\,\mathrm{mm}$.

- **Required tension steel area** (Equation 2.9):

$$A_s = \frac{M}{0.87 f_y z} = \frac{251.5 \cdot 10^6}{0.87 \cdot 500 \cdot 513} = 1127\,\mathrm{mm}^2$$

  Provide 2T25 + 1T20 ($A_s = 1296$ mm^2) as bottom steel.

- Check $\rho_{min} = 0.3\% < \rho = \frac{A_s}{b_w \cdot d} = \frac{1296}{300 \cdot 540} = 0.80\% < \rho_{max} = 2.5\% \rightarrow$ OK.

Design for Flexure at Interior Supports For supports B and C, the design moment is $M_{AB} = 490.3\,\mathrm{kNm}$. For negative, or hogging, moments the flange is under tension so the design of the section is that of a rectangular section ($b = b_w = 300$ mm):

- **Dimensionless Moment**:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{490.3 \cdot 10^6}{300 \cdot 540^2 \cdot 30} = 0.187 > 0.156 = K'$$

  Since $K > K'$, compression reinforcement is required.

- The strain of the compression steel is equal with

$$\varepsilon_s' = \varepsilon_{cu}\frac{x_{max} - d'}{x_{max}} = 0.0035\frac{0.5 \cdot 540 - 60}{0.5 \cdot 540} = 0.27\% > 0.217\% = \varepsilon_y$$

  hence the compression will yield.

- **Required compression steel area** (Equation 2.14):

$$A_s' = \frac{(K - K')f_{cu}bd^2}{0.87f_y(d - d')} = \frac{(0.187 - 0.156) \cdot 30 \cdot 300 \cdot 540^2}{0.87 \cdot 500 \cdot (540 - 54)} = 383\,\mathrm{mm}^2$$

  where $d' \approx 54$ mm (assuming cover of $30\,\mathrm{mm}$, link diameter of $12\,\mathrm{mm}$, and bar diameter of $12\,\mathrm{mm}$).

- **Required tension steel area** (Equation 2.15):

$$A_s = \frac{K' f_{cu} bd^2}{0.87f_y z} + A_s' = \frac{0.156 \cdot 30 \cdot 300 \cdot 540^2}{0.87 \cdot 500 \cdot 0.775 \cdot 540} + 383 = 2632\,\mathrm{mm}^2$$

  where $z = z_{min} = 0.775d = 0.775 \cdot 540 = 418.5\,\mathrm{mm}$ (for $K' = 0.156$).

- Provide 2T32 + 3T25 ($A_s = 3080\,\mathrm{mm}^2$) as top steel (tension) and 2T25 ($A_s' = 981\,\mathrm{mm}^2$) as bottom steel (compression).

- Check $\rho_{min} = 0.3\% < \rho = \frac{A_s}{b_w \cdot d} = \frac{3080}{300 \cdot 540} = 1.90\% < \rho_{max} = 2.5\% \rightarrow$ OK.

Design for Flexure at Interior Span (Negative Moment) For span BC (negative moment at supports), the design moment is $M_{BC} = 100.0\,\mathrm{kNm}$. Design as a rectangular section (with $b = b_w = 300$ mm):

- **Dimensionless Moment**:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{100.0 \cdot 10^6}{300 \cdot 540^2 \cdot 30} = 0.038 < 0.156 = K'$$

  Since $K < K'$, the section is singly reinforced.

- **Lever arm**:

$$z = 540 \cdot \left(0.5 + \sqrt{0.25 - \frac{0.038}{0.9}}\right) = 516\,\mathrm{mm}$$

  Check: $z > z_{\mathrm{max}} = 0.95d = 513\,\mathrm{mm}$, so use $z = z_{\mathrm{max}} = 513\,\mathrm{mm}$.

- **Required steel area** (Equation 2.9):

$$A_s = \frac{M}{0.87f_y z} = \frac{100.0 \cdot 10^6}{0.87 \cdot 500 \cdot 513} = 448\,\mathrm{mm}^2$$

  Provide 2T25 ($A_s = 981\,\mathrm{mm}^2$) as top steel, also used as hang-up bars.

- Check $\rho_{min} = 0.3\% < \rho = \frac{A_s}{b_w \cdot d} = \frac{981}{300 \cdot 540} = 0.61\% < \rho_{max} = 2.5\% \rightarrow$ OK.

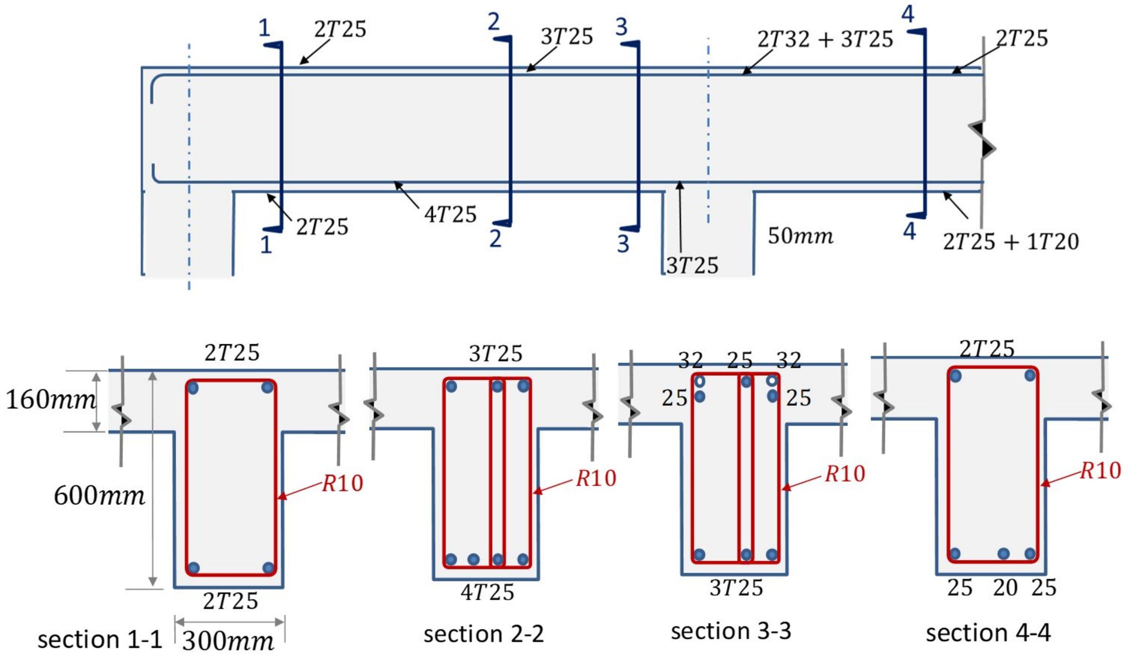

Figure 9.17: Longitudinal reinforcement along the beam and placement of longitudinal and shear reinforcement within the cross section.

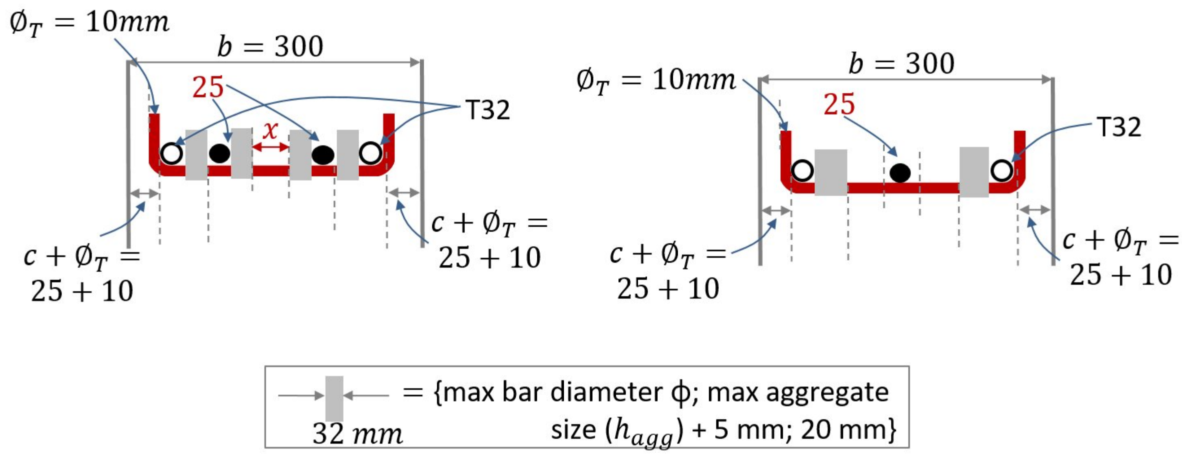

Figure 9.18: .

Detailing We cannot fit 2T32 + 3T25 in one layer within a beam width of $300\,\mathrm{mm}$ (see Figure 9.18 left). The practical clear distance between bars should be greater than $25\,\mathrm{mm}$ but must also exceed the bar diameter of $32\,\mathrm{mm}$. The remaining distance after placing 2T32 in the corners and 2T25 next to them, leaving the appropriate gaps ($32\,\mathrm{mm}$), is insufficient:

$$x = 300 - 2 \cdot (25 + 10) - 6 \cdot 32 - 2 \cdot 25 \quad \rightarrow \quad x = -12\,\mathrm{mm}$$

So we provide fit 2T32 + 1T25 in one layer (see Figure 9.18 right). and the remaining 2T25 in a second layer (see Figure 9.17 section 3-3). Placing the reinforcement into two layers reduces the moment resistance compared to a single layer of reinforcement. Capacity checking might be required to verify moment resistance with layered reinforcement.

9.2.3 Design for Shear

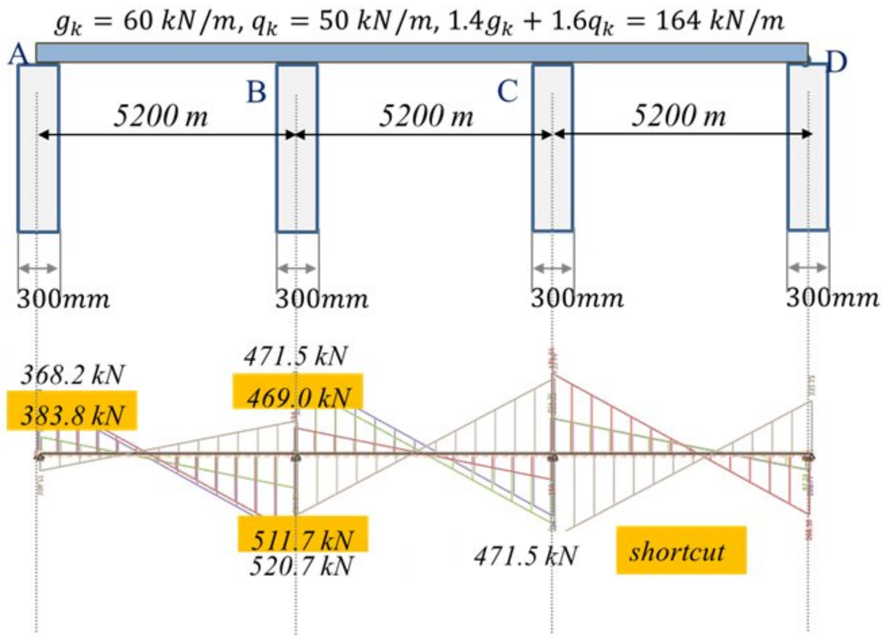

Figure 9.19: Shear force envelope. Assumed support width of 300mm.

Assume a support/column width of $300\,\mathrm{mm}$:

- **Maximum shear at face of support** (Equation A.2):

$$V_{\mathrm{max}} = V_{\mathrm{left}} - w \cdot (\text{support width}/2) = 520.1 - 164 \cdot 0.15 = 495.5\,\mathrm{kN}$$

- **Maximum shear stress check**:

$$v = \frac{V}{b_w d} = \frac{495.5 \cdot 10^3}{300 \cdot 540} = 3.06\,\mathrm{N/mm^2} < \min\left(7, 0.8\sqrt{f_{cu}} = 0.8\sqrt{30}\right) = 4.38\,\mathrm{N/mm^2}$$

  The section dimensions are sufficient.

- **Minimum links**:

$$\left(\frac{A_{sv}}{s_v}\right)_{\mathrm{min}} = \frac{0.4 b_w}{0.87 f_{yv}} = \frac{0.4 \cdot 300}{0.87 \cdot 250} = 0.552$$

  Provide R10@$275\,\mathrm{mm}$ minimum links ($\frac{A_{sv}}{s_v} = 0.57$).

Design for Shear at End Supports

For 4T25 bars ($A_s = 1963\,\mathrm{mm}^2$), $\dfrac{100A_s}{b_w d} = \dfrac{100 \cdot 1963}{300 \cdot 540} = 1.21$.
The design concrete shear stress v_c for $A_s = 1963\,\mathrm{mm}^2$ (from HKCC2013 Table 6.3) is:

$$v_c = 0.79\left(\frac{f_{cu}}{25}\right)^{\frac{1}{3}}\left(\frac{100A_s}{b_w d}\right)^{\frac{1}{3}}\left(\frac{400}{d}\right)^{\frac{1}{4}}\frac{1}{\gamma_m} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{\frac{1}{3}}(1.21)^{\frac{1}{3}}\left(\frac{400}{540}\right)^{\frac{1}{4}} = 0.72\,\mathrm{N/mm^2}$$

- **Shear at normal section (at d from support face)** (Equation A.2):

$$V_d = V_A - w \cdot \left(d + \frac{\text{support width}}{2}\right) = 368.2 - 164 \cdot (0.54 + 0.15) = 255.0\,\mathrm{kN}$$

- **Corresponding shear stress**:

$$v = \frac{V_d}{b_w d} = \frac{255.0 \cdot 10^3}{300 \cdot 540} = 1.57\,\mathrm{N/mm^2} > v_c + 0.4 = 1.12\,\mathrm{N/mm^2}$$

  Shear reinforcement is required, but suppose mild steel (R) is used.

- **Shear reinforcement**:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_w}{0.87 f_{yv}} = \frac{(1.57 - 0.72) \cdot 300}{0.87 \cdot 250} = 1.18$$

  Provide R10@130 mm links ($\frac{A_{sv}}{s_v} = 1.21$) closed to end supports (0–10 mm bars of $250\,\mathrm{N/mm^2}$ grade at $130\,\mathrm{mm}$ distance).

Design for Shear at Interior Supports

For 2T32 + 3T25 bars ($A_s = 3080\,\mathrm{mm}^2$), $\dfrac{100A_s}{b_w d} = \dfrac{100 \cdot 3080}{300 \cdot 540} = 1.90$.
The design concrete shear stress v_c for $A_s = 3080\,\mathrm{mm}^2$ (from HKCC2013 Table 6.3) is:

$$v_c = 0.79\left(\frac{f_{cu}}{25}\right)^{\frac{1}{3}}\left(\frac{100A_s}{b_w d}\right)^{\frac{1}{3}}\left(\frac{400}{d}\right)^{\frac{1}{4}}\frac{1}{\gamma_m} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{\frac{1}{3}}(1.90)^{\frac{1}{3}}\left(\frac{400}{540}\right)^{\frac{1}{4}} = 0.83\,\mathrm{N/mm^2}$$

- **Shear at normal section (at d from support face)**:

$$V_d = V_{\text{Bleft}} - w \cdot \left(d + \frac{\text{support width}}{2}\right) = 520.1 - 164 \cdot (0.54 + 0.15) = 406.9\,\mathrm{kN}$$

- **Corresponding shear stress**:

$$v = \frac{V_d}{b_w d} = \frac{406.9 \cdot 10^3}{300 \cdot 540} = 2.51\,\mathrm{N/mm^2} > v_c + 0.4 = 1.23\,\mathrm{N/mm^2}$$

  Shear reinforcement is required, but suppose mild steel (R) is used.

- **Shear reinforcement**:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_w}{0.87 f_{yv}} = \frac{(2.51 - 0.83) \cdot 300}{0.87 \cdot 250} = 2.32$$

  Provide R10@ $130\,\mathrm{mm}$ links in pairs ($\dfrac{A_{sv}}{s_v} = 2.42$).

Shear Resistance of Concrete Plus Minimum Links at Interior Supports

The shear resistance of the concrete plus minimum links is:

$$V_n = v_c b_w d + \left(\frac{A_{sv}}{s_v}\right) 0.87 f_{yv} d = 0.62 \cdot 300 \cdot 540 + 0.57 \cdot 0.87 \cdot 250 \cdot 540 = 167\,\mathrm{kN}$$

Since $V_d = 406.9\,\text{kN} > V_n = 167\,\text{kN}$, shear reinforcement beyond the minimum links is required over the distance from the face of the support:

$$s = \frac{V_d - V_n}{w} + d = \frac{406.9 - 167}{164} + 0.54 \approx 2.00\,\mathrm{m}$$

Optional Shear Design with Intermediate Links at Interior Supports

For long beams, between the maximum and minimum link regions, provide intermediate links:

- **Shear at 1.0 m from the support face** (Equation A.2):

$$V_{d,1} = V_d - w(1 - d) = 406.9 - 164 \cdot (1 - 0.54) = 331.5\,\mathrm{kN}$$

- **Corresponding shear stress**:

$$v = \frac{V_{d,1}}{b_w d} = \frac{331.5 \cdot 10^3}{300 \cdot 540} = 2.05\,\mathrm{N/mm^2}$$

- **Intermediate links**:

$$\left(\frac{A_{sv}}{s_v}\right) = \frac{(v - v_c) b_w}{0.87 f_{yv}} = \frac{(2.05 - 0.83) \cdot 300}{0.87 \cdot 250} = 1.68$$

  Provide R10@175 mm links in pairs ($\frac{A_{sv}}{s_v} = 1.79$) for 1.0 m from the support face.

Design for Shear at Interior Span

For 2T25 + 1T20 bars ($A_s = 1296\,\mathrm{mm}^2$), $\dfrac{100 A_s}{b_w d} = \dfrac{100 \cdot 1296}{300 \cdot 540} = 0.80$.
The design concrete shear stress v_c for $A_s = 1296\,\mathrm{mm}^2$ (from HKCC2013 Table 6.3) is:

$$v_c = 0.79 \left(\frac{f_{cu}}{25}\right)^{\frac{1}{3}} \left(\frac{100 A_s}{b_w d}\right)^{\frac{1}{3}} \left(\frac{400}{d}\right)^{\frac{1}{4}} \frac{1}{\gamma_m} = \frac{0.79}{1.25} \left(\frac{30}{25}\right)^{\frac{1}{3}} (0.80)^{\frac{1}{3}} \left(\frac{400}{540}\right)^{\frac{1}{4}} = 0.62\,\mathrm{N/mm^2}$$

- **Shear at the normal section (at d from support face)** (Equation A.2):

$$V_d = V_{\text{Bright}} - w \cdot \left(d + \frac{\text{support width}}{2}\right) = 471.5 - 164 \cdot (0.54 + 0.15) = 358.3\,\mathrm{kN}$$

- **Corresponding shear stress**:

$$v = \frac{V_d}{b_w d} = \frac{358.3 \cdot 10^3}{300 \cdot 540} = 2.21\,\mathrm{N/mm^2} > v_c + 0.4 = 1.02\,\mathrm{N/mm^2}$$

  Shear reinforcement is required, but suppose mild steel (R) is used.

- **Shear reinforcement**:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c) b_w}{0.87 f_{yv}} = \frac{(2.21 - 0.62) \cdot 300}{0.87 \cdot 250} = 2.19$$

  Provide R10@140 mm links in pairs ($\frac{A_{sv}}{s_v} = 2.62$).

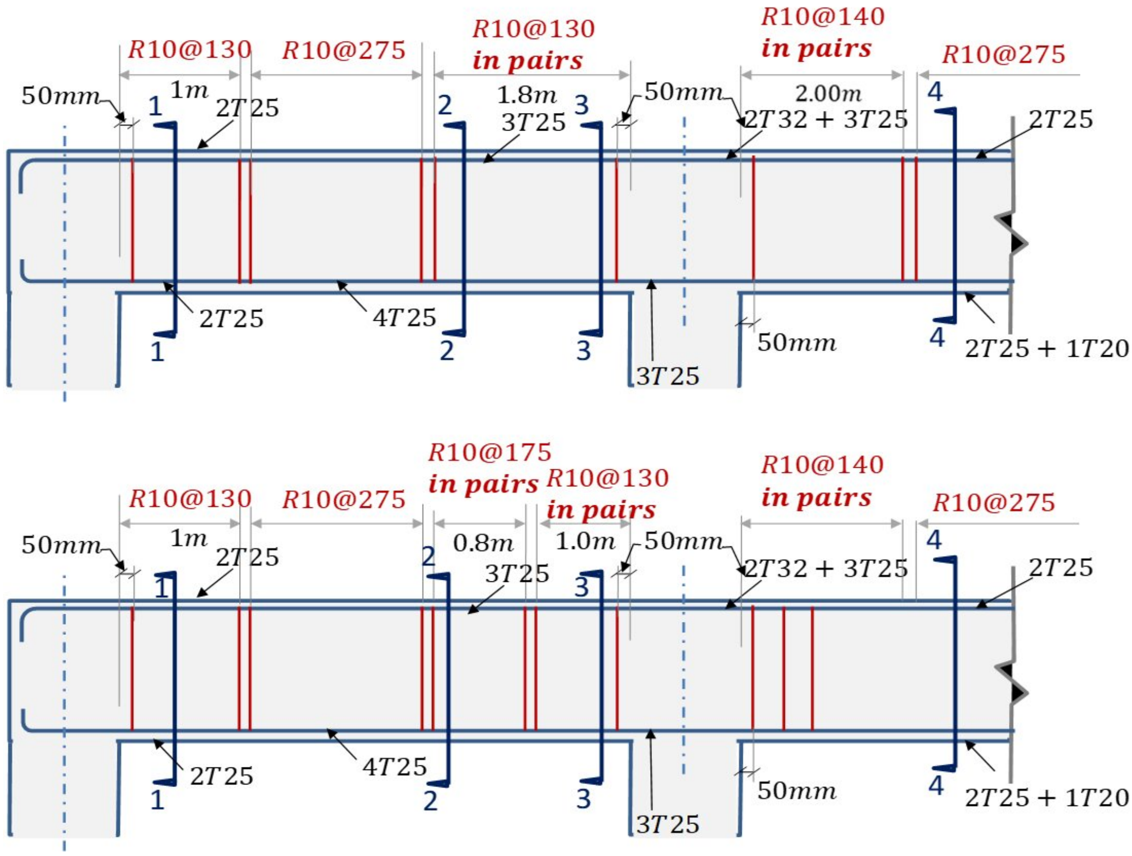

Figure 9.20: Shear reinforcement along the beam.

9.3 Tutorial #8: Complete Beam Design Example

Cross-sectional Dimensions: Section A-A' (Figure 9.21 right)

- Height: $h = 800\,\mathrm{mm}$
- Width: $b_w = 400\,\mathrm{mm}$

Material strengths:

- concrete $f_{cu} = 30\,\mathrm{MPa}$, longitudinal steel $f_y = 500\,\mathrm{MPa}$, shear links $f_{yv} = 250\,\mathrm{MPa}$

Design loads per area:

- dead $g_k = 7.5\,\mathrm{kN/m^2}$, live: $q_k = 11\,\mathrm{kN/m^2}$

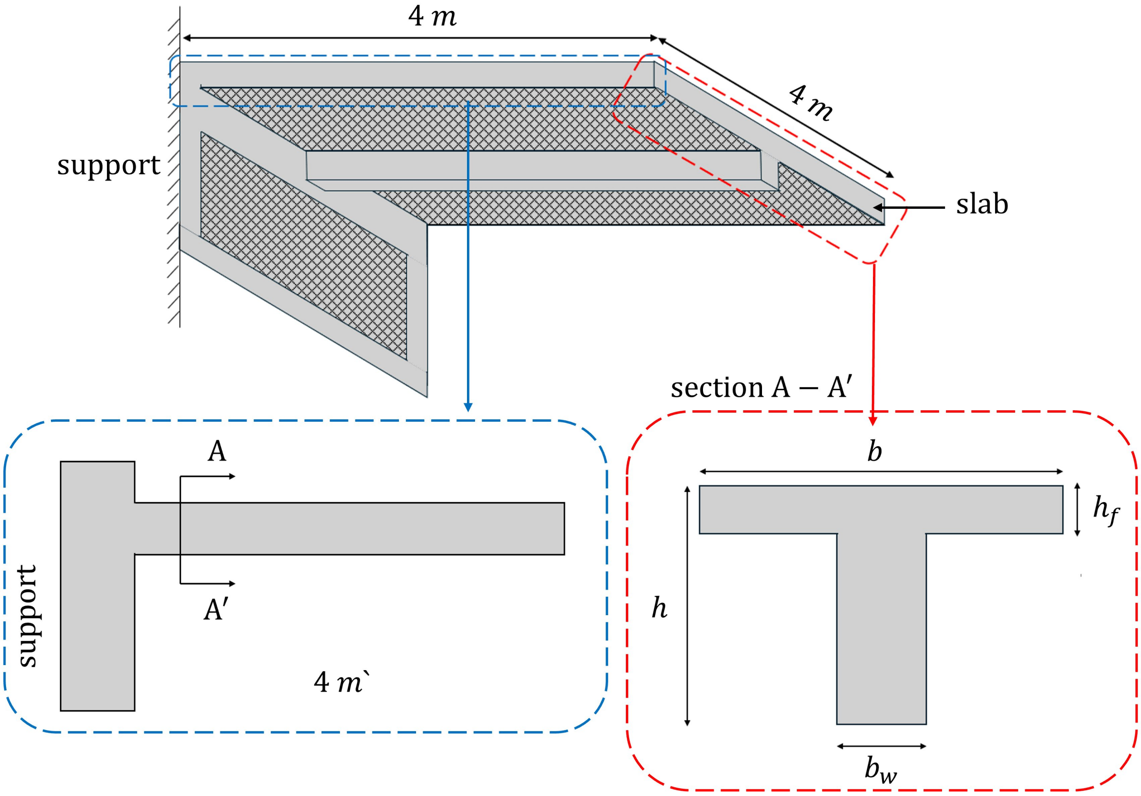

Figure 9.21: Sketch of the structure.

9.3.1 Structural Analysis

Loads and Moments

First, calculate the load per meter and then the ultimate design load:

$$G_k = g_k \cdot 4 = 7.5 \cdot 4 = 30\,\mathrm{kN/m}$$
$$Q_k = q_k \cdot 4 = 11 \cdot 4 = 44\,\mathrm{kN/m}$$
$$1.4G_k + 1.6Q_k = 1.4 \cdot 30 + 1.6 \cdot 44 = 112.4\,\mathrm{kN/m}$$

Using statics, we derive the shear and moment diagrams (see Figure 9.22) for the cantilever beam.

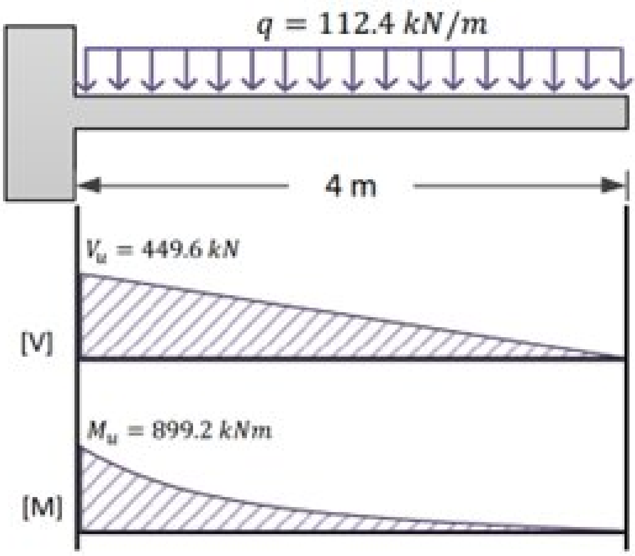

Figure 9.22: Shear and bending moment diagram of the cantilever beam.

9.3.2 Design for Flexure

Design Steps

1. Determine effective depth (d).
2. Check if singly or doubly reinforced.
3. Calculate required steel area and check reinforcement ratio.
4. Arrange and verify reinforcement.
5. SLS and detailing Requirements.
6. Ensure anchorage, bond, and lap.
7. Reinforcement curtailment

Determine effective depth (d)

$$d = h - (30 + \phi_T + \phi_L/2)$$

Where:

- $\phi_T = 10\,\mathrm{mm}$ (estimated diameter of shear links)
- $h = 800\,\mathrm{mm}$ (beam height)
- $\phi_T = 10\,\mathrm{mm}$ (estimated diameter of shear links)
- $\phi_L = 20\,\mathrm{mm}$ (estimated diameter of tension steel bars)
- Cover = $30\,\mathrm{mm}$

Calculation:

$$d = 800 - (30 + 10 + 20/2) = 750\,\mathrm{mm}$$

Singly or Doubly Reinforced

Calculate the (dimensionless) bending moment ratio K:

$$K = \frac{M_u}{bd^2 f_{cu}} = \frac{899.2 \cdot 10^6}{400 \cdot 750^2 \cdot 30} = 0.133$$

Since $K = 0.133 < K' = 0.156$, the section is **singly reinforced**.
Lever arm:

$$z = d\left(0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right) = 750\left(0.5 + \sqrt{0.25 - \frac{0.133}{0.9}}\right) = 614.5\,\mathrm{mm}$$

Check limits: $z_{\text{min}} = 0.775d = 581.3\,\mathrm{mm} < z < z_{\text{max}} = 0.95d = 712.5\,\mathrm{mm}$.

Required steel and steel ratio

Required tension steel:

$$A_s = \frac{M_u}{0.87 f_y z} = \frac{899.2 \cdot 10^6}{0.87 \cdot 500 \cdot 614.5} = 3363\,\mathrm{mm}^2$$

Select **8T25** (area $=3927\,\mathrm{mm}^2$) and check the pertinent reinforcement ratio:

$$\rho_{\text{min}} = 0.3\% < \rho = \frac{A_s}{b \cdot d} \cdot 100 = \frac{3927}{400 \cdot 750} = 1.31\% < 2.5\% = \rho_{\text{max}}$$

Steel ratio check is satisfied.

Arrangement

For construction, ensure **minimum spacing** (max aggregate size $h_{\text{agg}} = 20\,\mathrm{mm}$, Figure 6.2):

- Horizontal spacing requirement: $s_{\text{min,h}} = h_{\text{agg}} + 5 = 25\,\mathrm{mm}$
- Vertical spacing: $s_{\text{min,v}} = \frac{2h_{\text{agg}}}{3} \approx 15\,\mathrm{mm}$

With $50\,\mathrm{mm}$ cover on both sides:

$$\text{Bars per layer} = \frac{b_w - 2 \cdot 50}{\phi_L + 25} + 1 = \frac{400 - 100}{50} + 1 = 7$$

Only 7 bars fit in one layer, so use two layers of 4T25 (Figure 9.23).
Capacity checking for the 2-layer arrangement: from the **force equilibrium** in the cross-section we determine the stress block depth s:

$$0.45 f_{cu} bs = 0.87 f_y A_s \rightarrow 0.45 \cdot 30 \cdot 400 \cdot s = 0.87 \cdot 500 \cdot 3927$$

$$s = 316\,\mathrm{mm}$$

The **moment equilibrium** in the cross-section gives:

$$M_R = 0.87 f_y A_{s1}(d_1 - s/2) + 0.87 f_y A_{s2}(d_2 - s/2) \rightarrow$$

$$M_R = 0.87 \cdot 500 \cdot 1964[(750 - 316/2) + (720 - 316/2)] \cdot 10^{-6} \rightarrow$$

$$M_R = 985.9\,\mathrm{kNm} > 899.2\,\mathrm{kNm}$$

Therefore, the arrangement of the reinforcement into two layers of 4T25 provides sufficient moment resistance. Calculate the revised effective depth (for shear calculations later):

$$d = \frac{720 \cdot 1964 + 750 \cdot 1964}{3927} = 735\,\mathrm{mm}$$

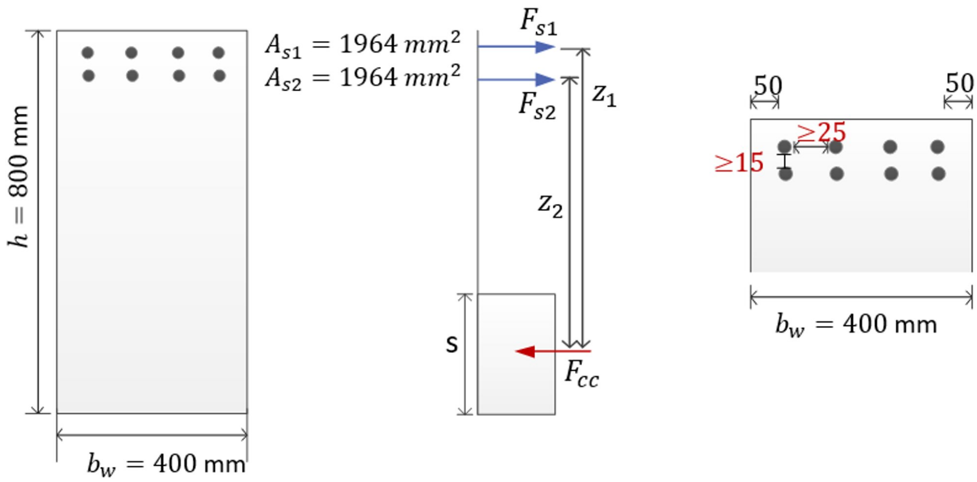

Figure 9.23: Forces in a 2-layer bar arrangement.

9.3.3 SLS and detailing Requirements

Check reinforcement area for SLS requirements on cracking and construction quality:

- **Maximum reinforcement ratio:**

$$\frac{A_s}{bh} = \frac{3927}{400 \cdot 800} \cdot 100 = 1.23\% < 4\%$$

- **Minimum reinforcement ratio:**

$$\frac{A_s}{bh} = 1.23\% > 0.26\%$$

- **Maximum clear spacing** between bars a_b (Figure 9.23 right):

$$a_b = \frac{b_w - (2 \cdot cover) - (4 \cdot \phi_L) - 2 \cdot \phi_T}{(\#of\ gaps\ between\ bars =)4 - 1} = \frac{400 - (2 \cdot 50) - (4 \cdot 25) - 2 \cdot 10}{3} = 60\,\mathrm{mm}$$

$25\,\mathrm{mm} \leq a_b \leq 150\,\mathrm{mm}$

- **Corner distance** a_c is the diagonal length from the surface of the tension rebar (the corner bars of the top layer) to the corner of the concrete cross-section:

$$a_c = \sqrt{50^2 + (800 - 750)^2} - \frac{25}{2} = 58.2\,\mathrm{mm} \leq 75\,\mathrm{mm}(\rightarrow \text{Check satisfied})$$

- **Side reinforcement**: Since $h = 800 > 750\,\mathrm{mm}$, we provide **side face** bars in the first $2h/3 = 533$ mm of the tension region of the beam (Figure 9.24 left). The maximum spacing $s_b \leq 250\,\mathrm{mm}$ determines the number of required side bars. Two layers are required in this case, since with two layers the spacing reduces to:

$$s_b = \frac{533 - (800 - 720)}{2} = 226\,\mathrm{mm} < 250\,\mathrm{mm}$$

  Bar diameter:

$$d > \sqrt{\frac{s_b b_w}{f_y}} = \sqrt{\frac{220 \cdot 400}{500}} = 13\,\mathrm{mm}$$

  Provide **2 layers of 2T16 at 220 mm spacing** (Figure 9.24).

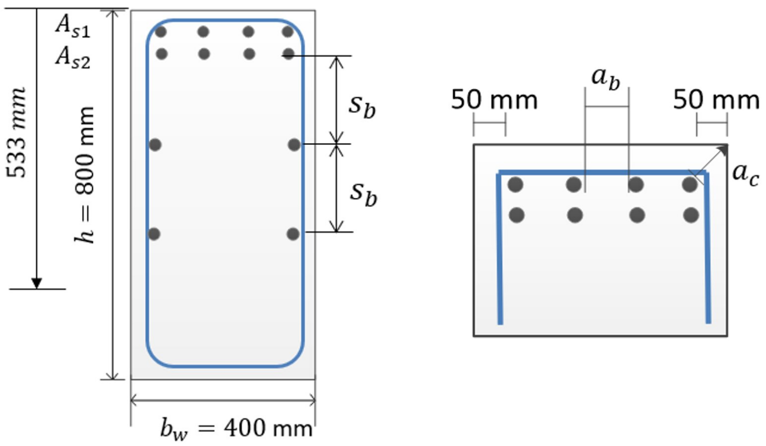

Figure 9.24: Left:Side face bar arrangement; Right: Spacing variables a_b and a_c.

- **Min compression steel**: for a flanged beam with the web under compression, Table 6.4:
$$\frac{100A_{sc}}{b_w h} \geq 0.2\%$$
Use **3T25 compression steel** see Figure 9.26.
- **Deflection**: span to effective depth ratio is $\frac{L}{d} = \frac{4000}{735} = 5.44 < 7$, acceptable.

9.3.4 Anchorage, Bond, and Lap

To ensure sufficient anchorage and bonding between steel rebars and concrete, we must provide sufficient **anchorage length**. We perform the calculations for a T25 bar in tension, but similar calculations must be repeated for all bars (including the compression steel). For a T25 steel rebar, ribbed, in tension $\beta = 0.5$ (Table 5.1). The design ultimate anchorage bond stress is:

$$f_{bu} = 0.5\sqrt{30} = 2.7\,\mathrm{N/mm^2}$$

The equation for anchorage clear span length is as follows:

$$l_b = \frac{0.87 f_y}{4 f_{bu}}\phi_e = \frac{0.87 \cdot 500}{4 \cdot 2.7} \cdot 25 = 1007\,\mathrm{mm}$$

Adding $d/2$ (see e.g., Figure 5.2):

$$l_b + \frac{d}{2} = 1007 + \frac{735}{2} = 1374.5\,\mathrm{mm}$$

Alternatively, we can estimate the anchorage length from Table 5.2, $K_A = 40$):

$$l_b = 40 \cdot 25 = 1000\,\mathrm{mm} \to l_b + \frac{735}{2} = 1367.5\,\mathrm{mm} \approx 1370\,\mathrm{mm}$$

The two values are practically equal, to say $1370\,\mathrm{mm}$.

If the support does not have sufficient space to house this anchorage, we must bend the bars to anchor them. For a 90° bend, the effective anchorage length is

$$l_e = 12\phi \to l_e = 12 \cdot 25 = 300\,\mathrm{mm}$$

We estimate the straight and bent length of the anchorage:

$$\text{Straight} = (l_b + d/2) - l_e = 1370\,\mathrm{mm} - 300\,\mathrm{mm} = 1070\,\mathrm{mm},$$

$$\text{Curved} = \frac{\pi}{2}(4 \cdot 25 + 25/2) = 216\,\mathrm{mm}$$

Total: $216 + 100 + 1070 = 1386\,\mathrm{mm}$.

9.3.5 Reinforcement curtailment

Figure 9.26 shows the final layout of longitudinal steel. Note that the full reinforcement $2 \times 4T25 = 8T25$ is not needed along the whole length of the beam, since the ultimate bending moment reduces rapidly towards the tip of the beam to zero.

For economy of material, we can curtail the reinforcement. Usually we can simply follow design curtailment guidelines. Instead, here we will do it analytically.

To this end, we first we calculate the moment capacity of half reinforcement (4T25). From the force equilibrium in the cross-section of Figure 9.25:

$$F_{cc} = F_{s1} \rightarrow 0.45 f_{cu} bs = 0.87 f_y A_{s1} \rightarrow 0.45 \cdot 30 \cdot 400 \cdot s = 0.87 \cdot 500 \cdot 1964 \rightarrow$$
$$158.2\,\mathrm{mm}$$

From the moment equilibrium for the same reinforcement (Figure 9.25):

$$M_R = 0.87 f_y A_{s1}(d_1 - s/2) = 0.87 \cdot 500 \cdot 1964 \cdot (750 - 158.2/2) \cdot 10^{-6} \rightarrow M_R = 573.2\,\mathrm{kNm}$$

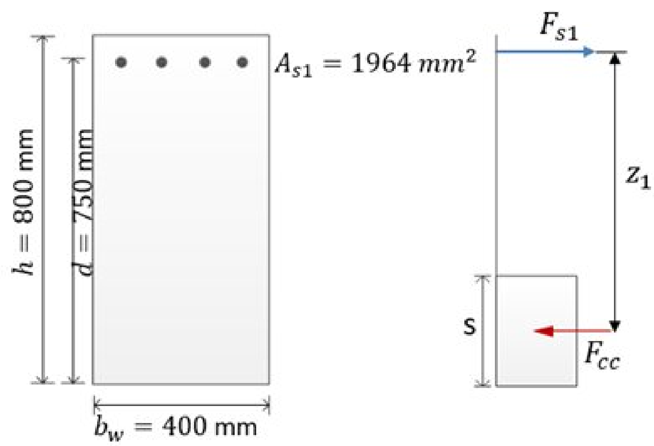

Figure 9.25: Forces in cross-section with only half the tension reinforcement.

From the bending moment diagram, we can identify the distance from the beam tip after which the moment drops below this M_R value:

$$M = \frac{ql^2}{2} = M_R \rightarrow 573.2 = \frac{112.4 l^2}{2} \rightarrow l = \sqrt{\frac{573.2 \cdot 2}{112.4}}\,\mathrm{m} \rightarrow l = 3.2\,\mathrm{m}$$

To ensure the bars can provide the calculated force at location $l = 3.2\,\mathrm{m}$ we must provide proper anchorage. Therefore, we need to continue the second layer of bars after $3200\,\mathrm{mm} - 1370\,\mathrm{mm} = 1830\,\mathrm{mm}$ from the end of the beam.

9.3.6 Design for Shear

Outline of design steps

1. Max shear stress check.
2. Check minimum shear reinforcement.
3. Shear reinforcement ratio in high shear zones.
4. Shear reinforcement ratio in non-high-shear zones.
5. Detailing.

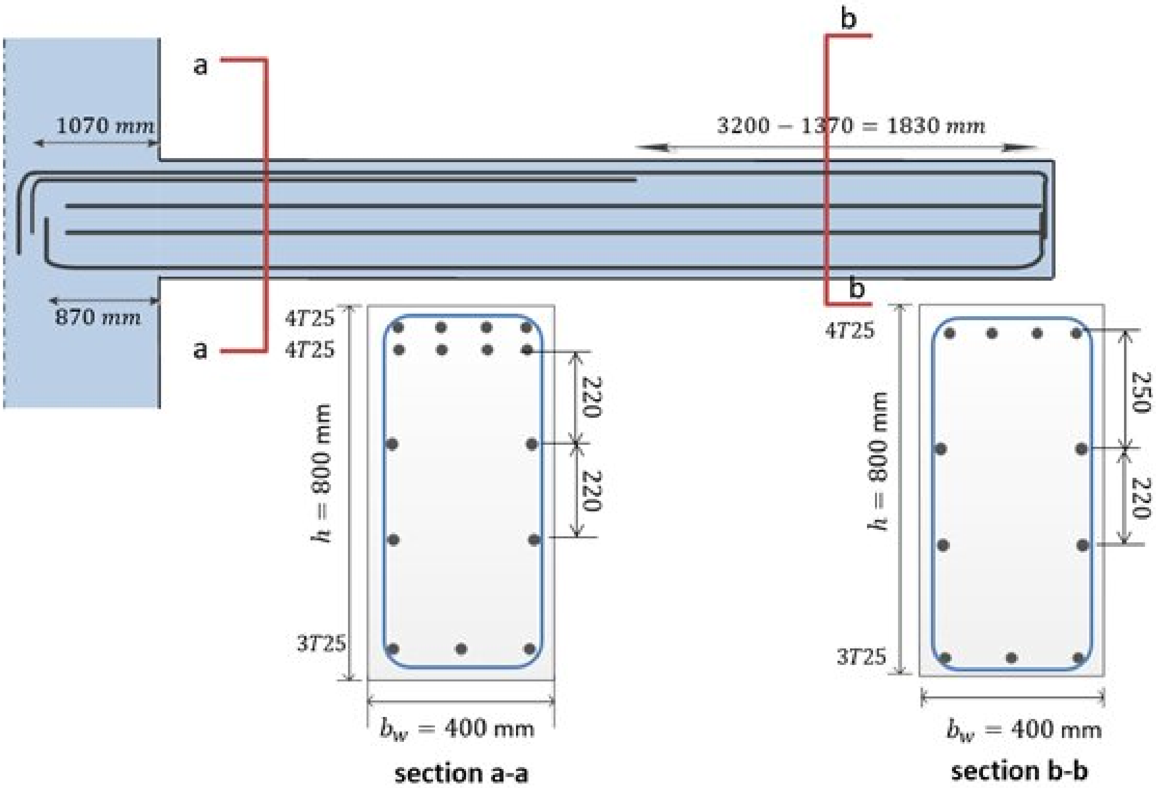

Figure 9.26: Complete flexure design.

Step 1: Maximum Shear Stress

First step is to ensure the maximum shear stress in the beam does not exceed the maximum allowable:

$$v = \frac{V_s}{b_v d} = \frac{449.6 \cdot 10^3}{400 \cdot 735} = 1.53\,\mathrm{MPa}$$

Check satisfied as:

$$v = 1.53\,\mathrm{MPa} < \min(0.8\sqrt{f_{cu}}, 7) = \min(0.8\sqrt{30} = 4.38, 7) = 4.38\,\mathrm{MPa}$$

Design concrete shear stress (Equation 3.3) for the calculated tension steel 8T25 ($A_s = 3927\,\mathrm{mm}^2$):

$$v_c = \frac{0.79}{\gamma_m}\left(\frac{f_{cu}}{25}\right)^{1/3}\left(\frac{100A_s}{b_v d}\right)^{1/3}\left(\frac{400}{d}\right)^{1/4} = \frac{0.79}{1.25}\left(\frac{30}{25}\right)^{1/3}\left(\frac{100 \cdot 3927}{400 \cdot 735}\right)^{1/3}\left(\frac{400}{735}\right)^{1/4}$$

where $\frac{100A_s}{b_v d}$ cannot be greater than 3, and $(\frac{400}{d})^{1/4}$ cannot be less than 1. Otherwise, $(\frac{400}{d})^{1/4} = 1$. Substituting we have:

$$v_c = \frac{0.79}{1.25} \cdot 1.2^{1/3} \cdot 1.33^{1/3} \cdot 1^{1/4} = 0.74 \rightarrow v_c = 0.74\,\mathrm{MPa}$$

From the shear force diagram (Figure 9.22), we can easily calculate the shear force at the nominal section, at distance d from the cantilever support:

$$V_s = 449.6 - 112.4 \cdot 0.735 = 359.6\,\mathrm{kN}$$

The demand in shear stress is:

$$v = \frac{359.6 \cdot 10^3}{400 \cdot 735} = 1.22\,\mathrm{MPa}$$

Step 2: Minimum Shear Reinforcement

Compare the shear concrete strength and the shear stress demand at the nominal section:

$$v = 1.22\,\mathrm{MPa} > v_c + 0.4 = 1.14\,\mathrm{MPa}$$

Minimum reinforcement is insufficient.

Step 3: Shear Reinforcement/Spacing

The required shear reinforcement ratio is:

$$\frac{A_{sv}}{s_v} = \frac{(v - v_c)b_v}{0.87 f_{yv}} = \frac{(1.22 - 0.74) \cdot 400}{0.87 \cdot 250} = 0.88$$

Use **R10@150**: a stirrup diameter of $10\,\mathrm{mm}$ at a spacing of $150\,\mathrm{mm}$.

Step 4: Minimum Links

Outside the high-shear zone, we can provide the minimum shear reinforcement ratio. For concrete strength below 40 N/mm^2 the minimum shear reinforcement ratio is (Table 6.2, clause 6.1.2.5) calculated for a shear stress of $v_r = 0.4$

$$\left(\frac{A_{sv}}{s_v}\right)_{\min} = \frac{v_r b_v}{0.87 f_{yv}} = \frac{0.4 \cdot 400}{0.87 \cdot 250} = 0.74\,\text{mm}$$

Use **R10@200**: i.e., a stirrup diameter of $10\,\mathrm{mm}$ at a spacing of $200\,\mathrm{mm}$. Max spacing: $0.75d = 551\,\mathrm{mm}$, satisfied.

The total shear capacity from concrete and minumum shear links is:

$$V_n = v_c b_v d + \left(\frac{A_{sv}}{s_v}\right)_{\min} 0.87 f_{yv} d \rightarrow$$

$$V_n = (0.74 \cdot 400 \cdot 735 + 0.74 \cdot 0.87 \cdot 250 \cdot 735) \cdot 10^{-3} = 335.9\,\mathrm{kN}$$

To determine the length of the high-shear zone, we identify the distance from the tip of the beam at which the shear force drops below the $V_n = 335.9$ kN value. From statics and our shear force diagram, it follows that :

$$x = \frac{335.9}{112.4} = 3.0\,\text{m from the tip of the beam}$$

Step 5: Detailing

Links required:

- Within the high-shear zone: R10 at $150\,\mathrm{mm}$ (8 links over $1\,\mathrm{m}$ from the support).
- Outside the high-shear zone: R10 at $200\,\mathrm{mm}$ (15 links over rest).

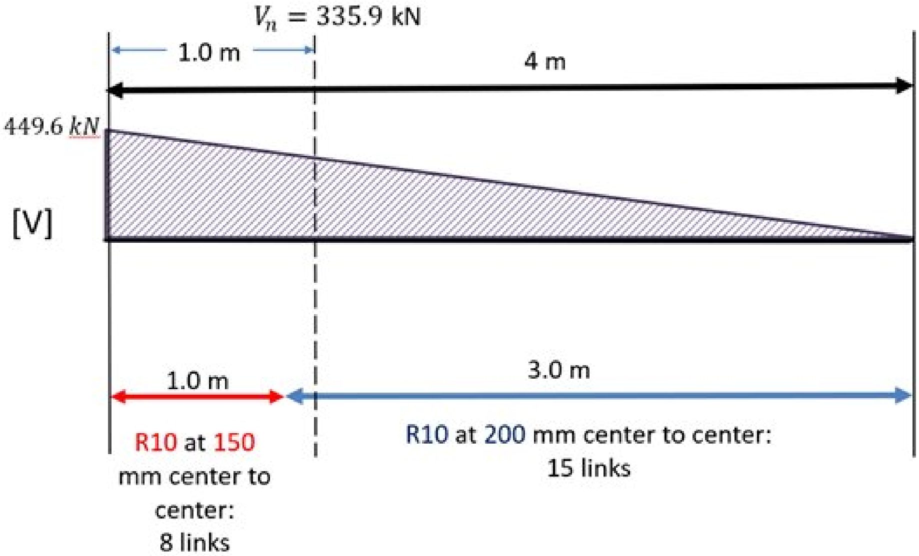

Figure 9.27: Shear link distribution.

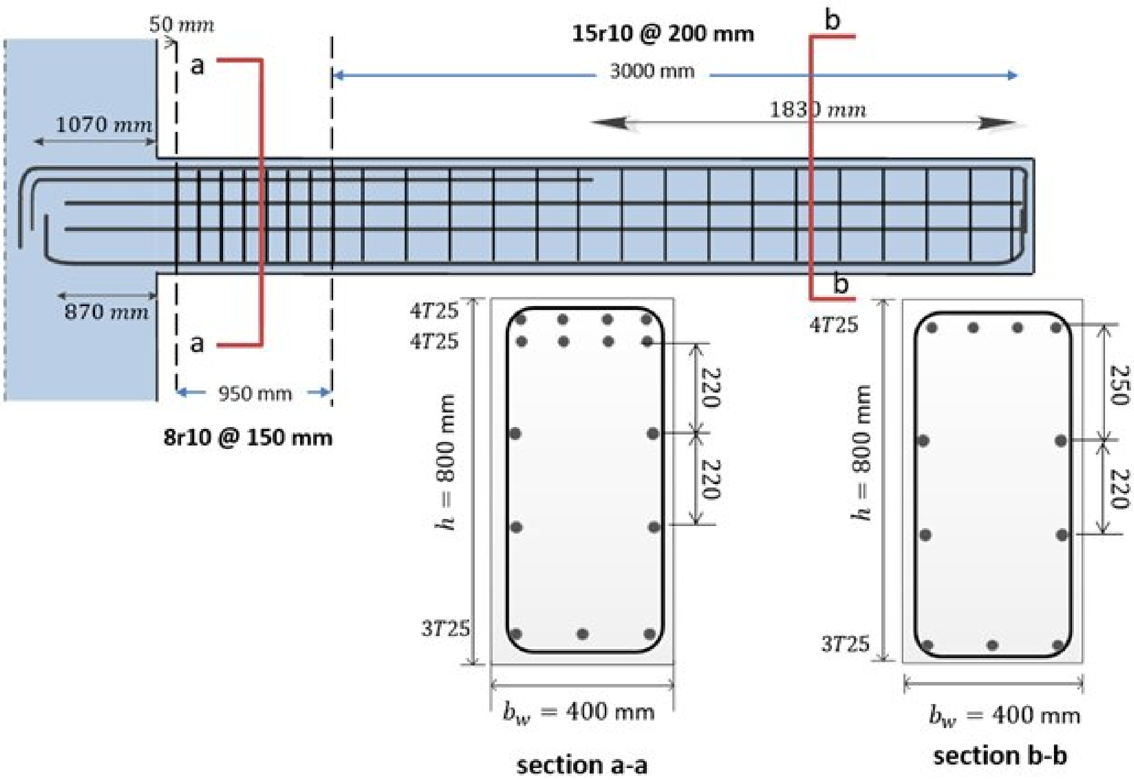

Figure 9.28: Final arrangement of reinforcement.

FAQs on Design Examples and Practical Aspects

This is a collection of frequently asked questions (FAQs) related to Design Examples and Practical Aspects, based on common student inquiries and clarifications provided during lectures.

Q1: Should the self-weight of reinforced concrete (R/C) be ignored in calculations?

No. The self-weight of R/C structural elements (approx. 24.5–25.0 kN/m^3) is a significant permanent load and must be included in the dead load (G_k) calculations.

Q2: Are there online tools available to assist with structural analysis?

Yes, several web-based solvers (such as https://structural-analyser.com/) can be used to verify beam reactions, shears, and moments, although hand calculations are required for exams and graded quizzes.

Q3: How do we handle indeterminate parameters like beam dimensions in preliminary design?

Designers make an "educated guess" for initial cross-section dimensions (b, h) based on span-to-depth ratios and architectural requirements. These dimensions are then verified against moment, shear, and deflection limits in subsequent design cycles.

Q4: Is there a standard beam width (b)?

While widths like 300 mm are common for standard residential/office beams, the final choice depends on the designer's decision, taking into account the required steel spacing and construction constraints.

Q5: How is the effective depth d calculated if $h = 550$ mm and cover is 50 mm?

The 50 mm allowance typically accounts for the cover plus the stirrup diameter. Assuming the center of the main reinforcement is 50 mm from the face, $d = h - 50 = 550 - 50 = 500$ mm.

Q6: Must reinforcement be placed symmetrically in a section?

Yes, symmetry (e.g., placing a larger bar in the center of the row) is preferred to avoid secondary torsional effects in the section and to ensure uniform stress distribution.

Q7: What is the minimum number of bars required in a beam section?

You need at least **four longitudinal bars** (two at the top and two at the bottom) to anchor the corners of the shear reinforcement (stirrups) and form a stable cage.

Q8: Can we use a combination of different bar diameters (e.g., $2T20 + 1T25$)?

Yes. This is often more economical and allows you to match the required A_s more precisely than using only one size. Ensure that the bars fit while maintaining minimum clear spacing and that the arrangement remains symmetric.

Q9: Can we use a single large diameter bar (e.g., 1 bar of 40mm) instead of multiple smaller bars?

It is not recommended. You must provide at least two tension rebars – one at each corner of the beam – to properly anchor the shear links (stirrups) and ensure the integrity of the reinforcement cage.

Q10: Why do we rarely use multiple rows of compression steel?

Steel is most efficient in tension, while concrete is most efficient in compression. If a section requires so much compression steel that multiple rows are needed, it is usually more efficient to increase the concrete section size rather than over-crowding the compression zone with steel.

Q11: Is there a risk of steel yielding during the construction phase?

Steel and concrete are supported by formwork while the concrete hardens. Yielding is highly unlikely once the target strength is reached and formwork is removed. However, certain scenarios like storing heavy machinery or construction waste on a structure under construction (refer to EN 1991-1-6) must be considered in specific load combinations.

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Appendix A

Statics Revision

A.1 Internal Forces Diagram - Shortcuts

Internal force diagrams, specifically shear force (V) and bending moment (M) diagrams, illustrate how internal forces vary along a beam, aiding in the design and analysis of structures.

The relationships between load intensity, shear force, and bending moment are foundational and can be analyzed examining the equilibrium of a infinitesimal length dx of material (see Fig A.1). Recall that:

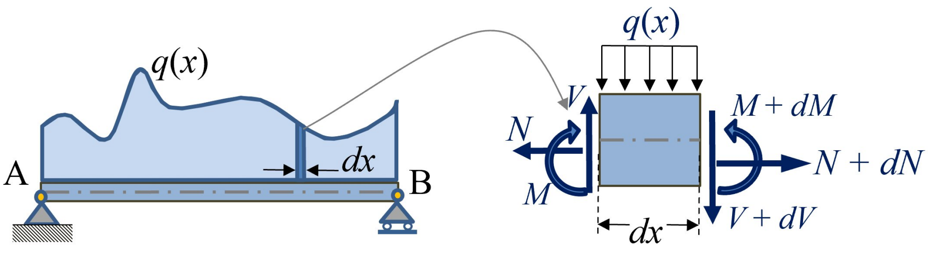

Figure A.1: Beam with a distributed load. The equilibrium of a infinitesimal length dx unveils the relationship between load intensity, shear force, and bending moment.

- The **shear force diagram** (V-diagram) is the integral of the distributed load intensity $q(x)$. Thus, the slope of the V-diagram at any point x is equal to the load intensity $q(x)$.
- The **bending moment diagram** (M-diagram) is the integral of the shear force $V(x)$. Consequently, the slope of the M-diagram at any point x is equal to the shear force $V(x)$.

These relationships are expressed mathematically as:

$$M_2 = M_1 + \int_1^2 V(x)\, dx \quad \text{(A.1)}$$

$$V_2 = V_1 - \int_1^2 q(x)\, dx \quad \text{(A.2)}$$

In Equation A.1, the change in bending moment from point 1 to point 2, $M_2 - M_1$, equals the area under the shear force diagram over the interval $[1, 2]$. In Equation A.2, the change in shear force, $V_2 - V_1$, is the negative of the area under the distributed load $q(x)$ over the same interval. The definite integral notation $\int_1^2$ specifies the limits, from location x_1 to x_2.

Conversely, for a beam with a distributed load (see Figure A.1), the differential relationships are:

$$\frac{dV(x)}{dx} = -q(x) \quad \text{(A.3)}$$

$$\frac{dM(x)}{dx} = V(x) \quad \text{(A.4)}$$

Equation A.3 indicates that the rate of change of shear force at any point x is the negative of the distributed load $q(x)$. Equation A.4 shows that the rate of change of the bending moment equals the shear force $V(x)$. These relationships allow engineers to construct shear and moment diagrams directly from loading conditions.

A.1.1 Practical Shortcuts

When constructing internal force diagrams, consider these practical guidelines:

- For a **uniformly distributed load** ($q(x) =$ constant), the shear force diagram is a straight line with slope $-q$, and the moment diagram is a parabola.
- For a **point load**, the shear force diagram shows a discontinuity (jump) equal to the load's magnitude, and the moment diagram exhibits a change in slope.
- For a **concentrated moment**, the moment diagram shows a jump, but the shear force diagram remains unaffected.

A.2 Analysis of Trusses

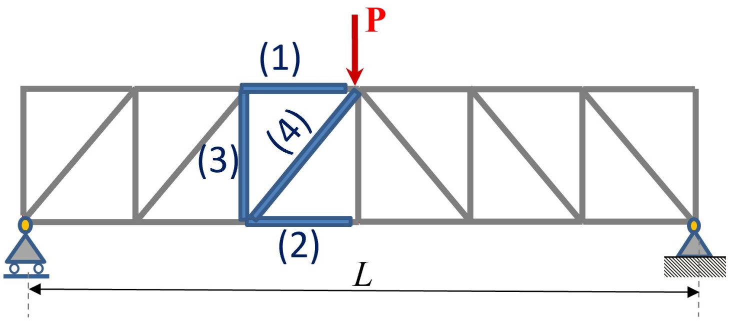

Figure A.2: Typical truss configuration with labeled members (1 to 4).

Trusses are frameworks of straight members connected at joints, designed to carry loads through axial forces (tension or compression). Analyzing trusses typically involves methods like the *method of joints* or the *method of sections* to determine member forces. However, qualitative approaches can also provide insights without solving the full system.

A.2.1 Qualitative Analysis of Member Forces

Consider a truss with members labeled 1 through 4, subjected to external loads and properly supported (see Figure A.2). A key question is: how can you determine if members are in *tension* ($F > 0$) or *compression* ($F < 0$) without fully solving the truss?

One effective method is the **beam analogy**. Imagine replacing the truss with a geometrically similar beam under the same loads and supports (see Figure A.3). By visualizing how the beam deforms, you can infer the state of truss members:

- **Top chord members** (e.g., member (1)) are typically in *compression*, analogous to the top fibers of a beam under bending, which shorten.
- **Bottom chord members** (e.g., member (2)) are usually in *tension*, similar to the bottom fibers of a beam, which elongate.

A.2.2 Method of Sections

For a more precise analysis, consider the method of sections, which involves cutting through the truss to expose internal forces and applying equilibrium equations. Suppose we cut through members 1, 2, and 4 and analyze the free-body diagram of one portion (see Figure A.4). Taking moments about strategic points simplifies the calculations. For example:

$$\Sigma M_A = 0 \rightarrow T_1 H - T_4 l = 0 \implies T_4 = \frac{H}{l} T_1 < 0 < 0 \tag{A.5}$$

$$\Sigma M_C = 0 \rightarrow A_y L_C + T_1 H = 0 \implies T_1 = -\frac{L_C}{H} A_y < 0 \tag{A.6}$$

$$\Sigma M_D = 0 \rightarrow A_y L_D - T_2 H = 0 \implies T_2 = \frac{L_D}{H} A_y > 0 \tag{A.7}$$

where, T_1, T_2, and T_4 are the internal forces in members 1, 2, and 4, respectively; H, l, L_C, and L_D are geometric dimensions; and A_y is the vertical reaction at support A. The signs of the member forces indicate tension (positive) or compression (negative), depending on

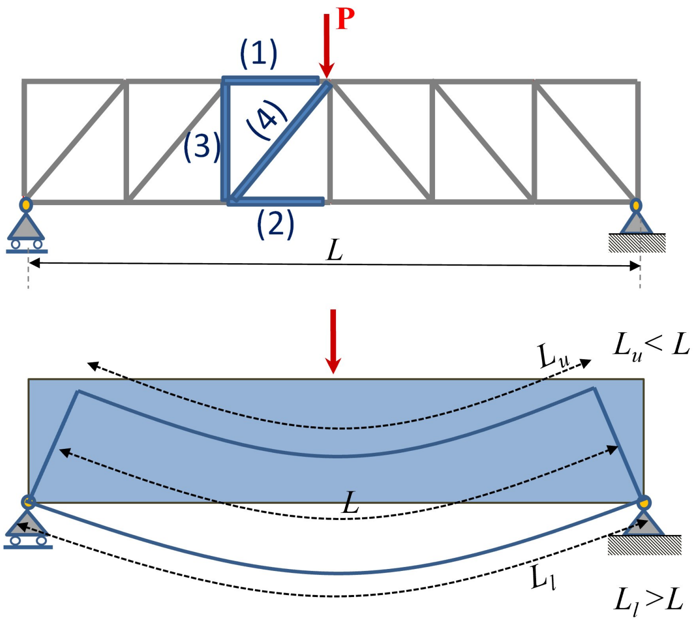

Figure A.3: Beam analogy for the truss, showing deformation under equivalent loads.

the assumed direction and the resulting values. Therefore, members T_1 and T_4 are under compression, and T_2 in tension. This method confirms the qualitative beam analogy for members like 2 (likely compression) and 4 (likely tension).

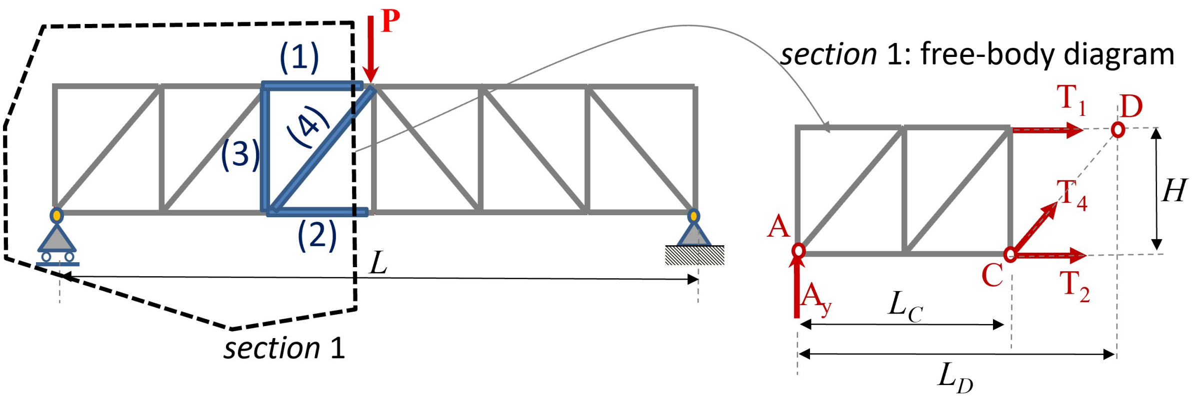

Figure A.4: Free-body diagram of a truss section, showing internal forces in members 1, 2, and 4.

FAQs on Structural Analysis and Design Aids

This is a collection of frequently asked questions (FAQs) related to tructural Analysis and Design Aids, based on common student inquiries and clarifications provided during lectures.

Q1: Do I need to draw influence lines?

Influence lines are a tool to identify which spans should receive "adverse" loading to produce the maximum effect at a specific location (e.g., max M_{AB} or max R_B). If you are already confident in the loading patterns (as discussed in Tutorial 3), you do not need to submit the influence lines themselves, but you must apply the correct load factors based on them.

Q2: Why is the shear force diagram mirrored or using different signs than SAP2000?

This is due to different sign conventions. SAP2000's default output may be mirrored compared to the convention taught in Tutorial 01. The design aid tables follow the lecture convention. Both are mathematically correct; simply be consistent and state your convention if it differs from the lecture notes.

Q3: How do I confirm the sign (+ or -) of a moment or reaction from the tables?

You should determine the general shape of your shear and bending moment diagrams (based on basic structural analysis principles) before applying the table formulas. This physical intuition will allow you to correctly apply the signs obtained from the numerical formulas.

Q4: Are the shear force diagrams for symmetric structures always mirrored?

Exactly. For a symmetric structure with symmetric loading, the shear force values on the right-hand side will be the mirrored (sign-flipped) equivalents of those on the left-hand side. This significantly simplifies the construction of the full shear force envelope.

Q5: In HW how do I determine if a load is adverse or beneficial for span AB or Support B?

You need to evaluate the cases that create the worst-case scenario for that specific location. Drawing the influence line for the moment at Support B and for the span AB will help you visually assign beneficial or adverse effect coefficients. Once you understand the influence line behavior, you apply the loads accordingly.